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Abstract
In this paper we introduce a geometric description of Lagrangian and Hamiltonian classical field theories on Lie algebroids in the framework of k-symplectic geometry. We discuss the relation between the Lagrangian and Hamiltonian descriptions through a convenient notion of Legendre transformation. The theory is a natural generalization of the standard one; in addition, other interesting examples are studied, in particular, systems with symmetry and Poisson-sigma models.

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1. Introduction

The Lie algebroid is a generalization of both the Lie algebra and the integrable distribution. The idea of using Lie algebroids in mechanics is due to Weinstein [47], who introduced a new geometric framework for Lagrangian mechanics. His formulation allows a geometric unified description of dynamical systems with a variety of different kinds of phase spaces: Lie groups, Lie algebras, Cartesian products of manifolds, or quotient manifolds (as in reduction theory, in which reduced phase spaces are not, in general, tangent or cotangent bundles). For a survey of further developments of this approach in relation to various mechanical problems, see [19].

One way of extending the theory to classical field theory is through the multisymplectic formalism [32, 33], which was independently developed by Tulczyjew's school in Warsaw (see, for instance, [17]), by García and Pérez–Rendón [10, 11] and by Goldschmidt and Sternberg [12], and has been revised by Martin [28, 29], Gotay *et al* [13] and Cantrijn *et al* [9], among others.

An alternative to the multisymplectic formalism is Günther's polysymplectic formalism [14] or equivalent presentations [36] involving k-symplectic structures defined independently by Awane [2, 3], Norris [34, 37–40] and de León *et al* [20, 22] (see also [21] and [23]). This approach is the generalization to certain kinds of field theory of the standard symplectic formalism of mechanics, the geometric framework for describing autonomous dynamical systems; the crucial device in Günther's formalism is the introduction of a vector-valued generalization of a symplectic form. It originally applied to theories with Lagrangians and Hamiltonians that do not depend on the base coordinates $t^1, \ldots, t^k$ (in many cases space-time coordinates), i.e. Lagrangian $L(q^i, v^i_A)$ and Hamiltonian $H(q^i, p^A_i)$ that depend only on the field coordinates q^i and on the partial derivatives of the field, v^i_A, or the corresponding momenta p^A_i. To treat more general situations we need to extend the formalism using k-cosymplectic geometry [24, 25].

The purpose of this paper is to extend the k-symplectic approach to first-order classical field theories on Lie algebroids. We present a geometric description of classical Lagrangian and Hamiltonian field theories on Lie algebroids, and we show the relation between them when the Lagrangian is hyperregular.

The paper is organized as follows. In section 2 we recall some basic elements of the k-symplectic approach to first-order classical field theories. In section 3 we recall some basic facts about Lie algebroids and their differential geometry, and the *prolongation of a Lie algebroid over a fibration*, which will be necessary for further developments. In section 4 the k-symplectic formalism is extended to Lie algebroids. Subsections 4.1 and 4.2 describe the extended Lagrangian and Hamiltonian formalisms, respectively, and in subsection 4.3 we define the Legendre transformation on Lie algebroids and establish the equivalence between the Lagrangian and Hamiltonian formalisms when the Lagrangian function is hyperregular. Finally in section 5 we display examples of the application of the theory to the Poisson-sigma model and first-order field theories with symmetries.

Throughout this paper, all manifolds and maps are C^∞, the Einstein summation convention is used, and k-tuples of elements are denoted by bold type.

2. Geometric preliminaries

In this section we recall some basic elements of the k-symplectic approach to classical field theories [14, 36, 41].

2.1. The tangent bundle of k^1-velocities of a manifold

Let Q be an n-dimensional differentiable manifold and $\tau_Q : TQ \to Q$ its tangent bundle. We denote by $T^1_k Q$ the Whitney sum $TQ \oplus \overset{k}{\cdots} \oplus TQ$ of k copies of TQ, with projection $\tau^k_Q : T^1_k Q \to Q$, $\tau^k_Q(v_{1_q}, \ldots, v_{k_q}) = q$, where $v_{A_q} \in T_q Q$, $A = 1, \ldots, k$. $T^1_k Q$ can be identified with the manifold $J^1_{\mathbf{0}}(\mathbb{R}^k, Q)$ of k^1-*velocities of* Q, that is 1-jets of maps $\sigma : \mathbb{R}^k \to Q$ with the source at $\mathbf{0} \in \mathbb{R}^k$, say

$$J^1_{\mathbf{0}}(\mathbb{R}^k, Q) \equiv TQ \oplus \overset{k}{\cdots} \oplus TQ$$
$$j^1_{\mathbf{0},q}\sigma \equiv (v_{1_q}, \ldots, v_{k_q}),$$

where $q = \sigma(\mathbf{0})$ and $v_{A_q} = \sigma_*(\mathbf{0})\left(\frac{\partial}{\partial t^A}\Big|_{\mathbf{0}}\right)$, $(t^1, \ldots, t^k)$ being the standard coordinates on $\mathbb{R}^k$. $T^1_k Q$ is called *the tangent bundle of k^1-velocities of Q* (see [35]).

If (q^i) are local coordinates on $U \subseteq Q$, then the induced local coordinates (q^i, v^i), $1 \leqslant i \leqslant n$, on $TU = \tau_Q^{-1}(U)$ are expressed by

$$q^i(v_q) = q^i(q), \qquad v^i(v_q) = v_q(q^i),$$

and the induced local coordinates (q^i, v^i_A), $1 \leqslant i \leqslant n$, $1 \leqslant A \leqslant k$, on $T^1_k U = (\tau^k_Q)^{-1}(U)$ are given by

$$q^i(v_{1q}, \ldots, v_{kq}) = q^i(q), \qquad v^i_A(v_{1q}, \ldots, v_{kq}) = v_{Aq}(q^i).$$

Let $f : M \to N$ be a differentiable map. The induced map $T^1_k f : T^1_k M \to T^1_k N$ defined by $T^1_k f(j^1_{\mathbf{0}}\sigma) = j^1_{\mathbf{0}}(f \circ \sigma)$ is called the *canonical prolongation* of f. Observe that

$$T^1_k f(v_{1q}, \ldots, v_{kq}) = (f_*(q)(v_{1q}), \ldots, f_*(q)(v_{kq})),$$

where $v_{1q}, \ldots, v_{kq} \in T_q Q$, $q \in Q$.

2.2. *k-vector fields and integral sections*

Let *M* be an arbitrary manifold.

Definition 2.1. *A section* $\mathbf{X} : M \longrightarrow T^1_k M$ *of the projection* τ^k_M *will be called a k-vector field on M.*

To give a *k*-vector field $\mathbf{X}$ is equivalent to giving a family of *k* vector fields $X_1, \ldots, X_k$, and we write $\mathbf{X} = (X_1, \ldots, X_k)$.

Definition 2.2. *An integral section of the k-vector field* $\mathbf{X} = (X_1, \ldots, X_k)$*, passing through a point* $x \in M$*, is a map* $\psi : U_{\mathbf{0}} \subset \mathbb{R}^k \to M$*, defined on some neighborhood* $U_{\mathbf{0}}$ *of* $\mathbf{0} \in \mathbb{R}^k$*, such that* $\psi(\mathbf{0}) = x$*, and*

$$\psi_*(\mathbf{t}) \left(\left. \frac{\partial}{\partial t^A} \right|_{\mathbf{t}} \right) = X_A(\psi(\mathbf{t})) \qquad \text{for every} \quad \mathbf{t} \in U_{\mathbf{0}}, 1 \leqslant A \leqslant k,$$

or equivalently $\psi(\mathbf{0}) = x$ *and* ψ *satisfies* $\mathbf{X} \circ \psi = \psi^{(1)}$*, where* $\psi^{(1)}$ *is the first prolongation of* ψ *to* $T^1_k M$*, defined by*

$$\begin{array}{rccl} \psi^{(1)} : & U_0 \subset \mathbb{R}^k & \longrightarrow & T^1_k M \\ & \mathbf{t} & \longrightarrow & \psi^{(1)}(\mathbf{t}) = j^1_{\mathbf{0}}\psi_{\mathbf{t}} \equiv \left(\psi_*(\mathbf{t}) \left(\left. \frac{\partial}{\partial t^1} \right|_{\mathbf{t}} \right), \ldots, \psi_*(\mathbf{t}) \left(\left. \frac{\partial}{\partial t^k} \right|_{\mathbf{t}} \right) \right), \end{array}$$

where $\psi_{\mathbf{t}}(\mathbf{s}) = \psi(\mathbf{t} + \mathbf{s})$.

A k-vector field $\mathbf{X} = (X_1, \ldots, X_k)$ *on M is said to be integrable if there is an integral section that passes through every point of M.*

Remark 2.3. In the *k*-symplectic formalism, the solutions of field equations are the integral sections of *k*-vector fields. In the case $k = 1$, this definition coincides with the classical definition of the integral curve of a vector field.

In a local coordinate system, if $\psi(\mathbf{t}) = (\psi^i(\mathbf{t}))$, then one has

$$\psi^{(1)}(\mathbf{t}) = \left(\psi^i(\mathbf{t}), \left. \frac{\partial \psi^i}{\partial t^A} \right|_{\mathbf{t}} \right), \qquad 1 \leqslant A \leqslant k, \quad 1 \leqslant i \leqslant n, \tag{2.1}$$

and ψ is an integral section of $(X_1, \ldots, X_k)$, where $X_A = X^i_A \frac{\partial}{\partial q^i}$ if and only if

$$\frac{\partial \psi^i}{\partial t^A} = X^i_A \circ \psi, \qquad 1 \leqslant A \leqslant k, \quad 1 \leqslant i \leqslant n. \tag{2.2}$$

2.3. The cotangent bundle of k^1-covelocities of a manifold

Let Q be a differentiable manifold of dimension n and $\pi_Q : T^*Q \to Q$ its cotangent bundle. Denote by $(T^1_k)^*Q = T^*Q \oplus \stackrel{k}{\cdots} \oplus T^*Q$ the Whitney sum of k copies of T^*Q with the projection map $\pi^k_Q : (T^1_k)^*Q \to Q, \pi^k_Q(\alpha_{1_q}, \ldots, \alpha_{k_q}) = q$. The manifold $(T^1_k)^*Q$ can be canonically identified with the vector bundle $J^1(Q, \mathbb{R}^k)_{\mathbf{0}}$ of k^1-covelocities of the manifold Q, the manifold of 1-jets of maps $\sigma : Q \to \mathbb{R}^k$ with the target at $\mathbf{0} \in \mathbb{R}^k$ and the projection map $\pi^k_Q : J^1(Q, \mathbb{R}^k)_{\mathbf{0}} \to Q, \pi^k_Q(j^1_{q,\mathbf{0}}\sigma) = q$, that is

$$J^1(Q, \mathbb{R}^k)_{\mathbf{0}} \equiv T^*Q \oplus \stackrel{k}{\cdots} \oplus T^*Q$$
$$j^1_{q,\mathbf{0}}\sigma \equiv (\mathrm{d}\sigma_1(q), \ldots, \mathrm{d}\sigma_k(q)),$$

where $\sigma_A = pr_A \circ \sigma : Q \longrightarrow \mathbb{R}$ is the Ath component of σ and $pr_A : \mathbb{R}^k \to \mathbb{R}$ are the canonical projections, $1 \leqslant A \leqslant k$. For this reason, $(T^1_k)^*Q$ is also called *the bundle of k^1-covelocities of the manifold Q*.

If (q^i) are local coordinates on $U \subseteq Q$, then the induced local coordinates (q^i, p_i) on $T^*U = (\pi_Q)^{-1}(U)$ are given by

$$q^i(\alpha_q) = q^i(q), \qquad p_i(\alpha_q) = \alpha_q\left(\left.\frac{\partial}{\partial q^i}\right|_q\right), \qquad 1 \leqslant i \leqslant n,$$

and the induced local coordinates (q^i, p^A_i) on $(T^1_k)^*U = (\pi^k_Q)^{-1}(U)$ are

$$q^i(\alpha_{1_q}, \ldots, \alpha_{k_q}) = q^i(q), \quad p^A_i(\alpha_{1_q}, \ldots, \alpha_{k_q}) = \alpha_{A_q}\left(\left.\frac{\partial}{\partial q^i}\right|_q\right), \quad 1 \leqslant i \leqslant n, \quad 1 \leqslant A \leqslant k.$$

We can endow $(T^1_k)^*Q$ with a k-symplectic structure given by the family $(\omega^1, \ldots, \omega^k; V = \ker T\pi^k_Q)$ where each ω^A is the 2-form given by

$$\omega^A = (\pi^{k,A}_Q)^*\omega_Q, \qquad 1 \leqslant A \leqslant k,$$

$\pi^{k,A}_Q : (T^1_k)^*Q \to T^*Q$ being the canonical projection onto the Ath copy of T^*Q in $(T^1_k)^*Q$ and ω_Q the canonical symplectic form on T^*Q. In local coordinates, $\omega^A = dq^i \wedge dp^A_i$ [2, 3, 36, 41].

3. Lie algebroids

In this section we present some basic facts about Lie algebroids, including features of the associated differential calculus and results on Lie algebroid morphisms that will be necessary. For further information on groupoids and Lie algebroids, and their roles in differential geometry, see [4, 15, 26, 27].

3.1. Lie algebroid: definition

Let E be a vector bundle of rank m over a manifold Q of dimension n, and let $\tau : E \to Q$ be the vector bundle projection. Denote by $\mathrm{Sec}(E)$ the $C^\infty(Q)$-module of sections of τ. A *Lie algebroid structure* $([\![\cdot,\cdot]\!]_E, \rho_E)$ on E is a Lie bracket $[\![\cdot,\cdot]\!]_E : \mathrm{Sec}(E) \times \mathrm{Sec}(E) \to \mathrm{Sec}(E)$ on the space $\mathrm{Sec}(E)$, together with an *anchor map* $\rho_E : E \to TQ$ and its identically denoted induced $C^\infty(Q)$-module homomorphism $\rho_E : \mathrm{Sec}(E) \to \mathfrak{X}(Q)$, such that the *compatibility condition*

$$[\![\sigma_1, f\sigma_2]\!]_E = f[\![\sigma_1, \sigma_2]\!]_E + (\rho_E(\sigma_1)f)\sigma_2$$

holds for all smooth functions f on Q and sections σ_1, σ_2 of E (here $\rho_E(\sigma_1)$ is the vector field on Q given by $\rho_E(\sigma_1)(q) = \rho_E(\sigma_1(q))$). The triple $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ is called a *Lie algebroid over Q*. From the compatibility condition and the Jacobi identity, it follows that $\rho_E : \mathrm{Sec}(E) \to \mathfrak{X}(Q)$ is a homomorphism between the Lie algebras $(\mathrm{Sec}(E), [\![\cdot,\cdot]\!]_E)$ and $(\mathfrak{X}(Q), [\cdot,\cdot])$. The following are examples of Lie algebroids.

(i) *Real Lie algebras of finite dimension*. Any real Lie algebra of finite dimension is a Lie algebroid over a single point.

(ii) *The tangent bundle*. If TQ is the tangent bundle of a manifold Q, then the triple $(TQ, [\cdot,\cdot], id_{TQ})$ is a Lie algebroid over Q, where $id_{TQ} : TQ \to TQ$ is the identity map.

(iii) A less immediate example of a Lie algebroid may be constructed as follows. Let $\pi : P \to Q$ be a principal bundle with structural group G. Denote by $\Phi : G \times P \to P$ the free action of G on P and by $T\Phi : G \times TP \to TP$ the tangent action of G on TP. Then the sections of the quotient vector bundle $\tau_{P|G} : TP/G \to Q = P/G$ may be identified with the vector fields on P that are invariant under the action Φ. Since every G-invariant vector field on P is π-projectable and the standard Lie bracket on vector fields is closed with respect to G-invariant vector fields, we can define a Lie algebroid structure on TP/G. The resultant Lie algebroid over Q is called *the Atiyah (gauge) algebroid associated with the principal bundle* $\pi : P \to Q$ [19, 26].

Throughout this paper, the role of the Lie algebroid is to stand in for the tangent bundle of Q. In this way, one regards an element e of E as a generalized velocity, and the actual velocity v is obtained when we apply the anchor map to e, i.e. $v = \rho_E(e)$.

Let $(q^i)_{i=1}^n$ be the local coordinates on Q and $\{e_\alpha\}_{1\leqslant\alpha\leqslant m}$ a local basis of sections of τ. Given $e \in E$ such that $\tau(e) = q$, we can write $e = y^\alpha(e)e_\alpha(q) \in E_q$, i.e. each section σ is given locally by $\sigma|_U = y^\alpha e_\alpha$ and the coordinates of e are $(q^i(e), y^\alpha(e))$. A Lie algebroid structure on Q is determined locally by a set of local *structure functions* $\rho^i_\alpha, \mathcal{C}^\gamma_{\alpha\beta}$ on Q that are defined by

$$\rho_E(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial q^i}, \qquad [\![e_\alpha, e_\beta]\!]_E = \mathcal{C}^\gamma_{\alpha\beta} e_\gamma, \tag{3.1}$$

and satisfy the relations

$$\sum_{\mathrm{cyclic}(\alpha,\beta,\gamma)} \left(\rho^i_\alpha \frac{\partial \mathcal{C}^\nu_{\beta\gamma}}{\partial q^i} + \mathcal{C}^\nu_{\alpha\mu}\mathcal{C}^\mu_{\beta\gamma} \right) = 0, \qquad \rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial q^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial q^j} = \rho^i_\gamma \mathcal{C}^\gamma_{\alpha\beta}. \tag{3.2}$$

These relations, which are a consequence of the compatibility condition and Jacobi's identity, are usually called *the structure equations* of the Lie algebroid E.

3.2. Exterior differential

A Lie algebroid structure on E allows us to define *the exterior differential of E*, d^E: $\mathrm{Sec}(\bigwedge^l E^*) \to \mathrm{Sec}(\bigwedge^{l+1} E^*)$ as follows:

$$\begin{aligned} \mathrm{d}^E\mu(\sigma_1,\ldots,\sigma_{l+1}) = {}& \sum_{i=1}^{l+1}(-1)^{i+1}\rho_E(\sigma_i)\mu(\sigma_1,\ldots,\widehat{\sigma_i},\ldots,\sigma_{l+1}) \\ & + \sum_{i<j}(-1)^{i+j}\mu([\sigma_i,\sigma_j]_E,\sigma_1,\ldots,\widehat{\sigma_i},\ldots,\widehat{\sigma_j},\ldots\sigma_{l+1}), \end{aligned} \tag{3.3}$$

for $\mu \in \mathrm{Sec}(\bigwedge^l E^*)$ and $\sigma_1,\ldots,\sigma_{l+1} \in \mathrm{Sec}(E)$. It follows that d^E is a cohomology operator, that is $(\mathrm{d}^E)^2 = 0$.

In particular, if $f : Q \to \mathbb{R}$ is a smooth real function, then $d^E f(\sigma) = \rho_E(\sigma)f$, for $\sigma \in \mathrm{Sec}(E)$. Locally, the exterior differential is determined by

$$d^E q^i = \rho^i_\alpha e^\alpha \qquad \text{and} \qquad d^E e^\gamma = -\tfrac{1}{2}\mathcal{C}^\gamma_{\alpha\beta} e^\alpha \wedge e^\beta,$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$.

The usual Cartan calculus extends to the case of Lie algebroids: for every section σ of E we have a derivation i_σ (contraction) of degree -1 and a derivation $\mathcal{L}_\sigma = i_\sigma \circ d + d \circ i_\sigma$ (the Lie derivative) of degree 0; for more details, see [26, 27].

3.3. Morphisms

Let $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ and $(E', [\![\cdot,\cdot]\!]_{E'}, \rho_{E'})$ be two Lie algebroids over Q and Q', respectively, and suppose that $\Phi = (\overline{\Phi}, \underline{\Phi})$ is a vector bundle map, that is $\overline{\Phi} : E \to E'$ is a fiberwise linear map over $\underline{\Phi} : Q \to Q'$. The pair $(\overline{\Phi}, \underline{\Phi})$ is said to be a *Lie algebroid morphism* if

$$d^E(\Phi^*\sigma') = \Phi^*(d^{E'}\sigma'), \qquad \text{for all } \sigma' \in \mathrm{Sec}(\textstyle\bigwedge^l (E')^*) \text{ and for all } l. \tag{3.4}$$

Here $\Phi^*\sigma'$ is the section of the vector bundle $\bigwedge^l E^* \to Q$ defined (for $l > 0$) by

$$(\Phi^*\sigma')_q(e_1, \ldots, e_l) = \sigma'_{\underline{\Phi}(q)}(\overline{\Phi}(e_1), \ldots, \overline{\Phi}(e_l)), \tag{3.5}$$

for $q \in Q$ and $e_1, \ldots, e_l \in E_q$. In particular, when $Q = Q'$ and $\underline{\Phi} = id_Q$ then (3.4) holds if and only if

$$[\![\overline{\Phi}\circ\sigma_1, \overline{\Phi}\circ\sigma_2]\!]_{E'} = \overline{\Phi}[\![\sigma_1, \sigma_2]\!]_E, \qquad \rho_{E'}(\overline{\Phi}\circ\sigma) = \rho_E(\sigma), \qquad \text{for} \quad \sigma, \sigma_1, \sigma_2 \in \mathrm{Sec}(E).$$

Let (q^i) be a local coordinate system on Q and $(\bar{q}^{\bar{i}})$ a local coordinate system on Q'. Let $\{e_\alpha\}$ and $\{\bar{e}_{\bar{\alpha}}\}$ be local bases of sections of E and E', respectively, and $\{e^\alpha\}$ and $\{\bar{e}^{\bar{\alpha}}\}$ their respective dual bases. The vector bundle map Φ is determined by the relations $\Phi^*\bar{q}^{\bar{i}} = \phi^{\bar{i}}(q)$ and $\Phi^*\bar{e}^{\bar{\alpha}} = \phi^{\bar{\alpha}}_\beta e^\beta$ for certain local functions $\phi^{\bar{i}}$ and $\phi^{\bar{\alpha}}_\beta$ on Q. In this coordinate system $\Phi = (\overline{\Phi}, \underline{\Phi})$ is a Lie algebroid morphism if and only if

$$(\rho_E)^j_\alpha \frac{\partial \phi^{\bar{i}}}{\partial q^j} = (\rho_{E'})^{\bar{i}}_{\bar{\beta}} \phi^{\bar{\beta}}_\alpha, \qquad \phi^{\bar{\beta}}_\gamma \mathcal{C}^\gamma_{\alpha\delta} = \left((\rho_E)^i_\alpha \frac{\partial \phi^{\bar{\beta}}_\delta}{\partial q^i} - (\rho_E)^i_\delta \frac{\partial \phi^{\bar{\beta}}_\alpha}{\partial q^i} \right) + \bar{\mathcal{C}}^{\bar{\beta}}_{\bar{\theta}\bar{\sigma}} \phi^{\bar{\theta}}_\alpha \phi^{\bar{\sigma}}_\delta, \tag{3.6}$$

where the $(\rho_E)^i_\alpha, \mathcal{C}^\alpha_{\beta\gamma}$ are the structure functions on E and the $(\rho_{E'})^{\bar{i}}_{\bar{\alpha}}, \bar{\mathcal{C}}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}$ are the structure functions on E'.

For more about the concept of Lie algebroids morphism, see for instance [8, 15, 32, 33].

3.4. The prolongation of a Lie algebroid over a fibration

In this subsection we recall a particular kind of Lie algebroid that will be used later (see [8, 15, 19, 30] for more details).

If $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ is a Lie algebroid over a manifold Q and $\pi : P \to Q$ is a fibration, then

$$\tilde{\tau}_P : \mathcal{T}^E P = \bigcup_{p\in P} \mathcal{T}^E_p P \to P,$$

where

$$\mathcal{T}^E_p P = \{(e, v_p) \in E_{\pi p} \times T_p P | \rho_E(e) = T_p\pi(v_p)\}$$

is a Lie algebroid called the prolongation of the Lie algebroid $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ or the inverse-image Lie algebroid; see for instance [15, 19]. The anchor map of this Lie algebroid is $\rho^\pi : \mathcal{T}^E P \to TP$, $\rho^\pi(e, v_p) = v_p$. In this paper we consider two particular Lie algebroid prolongations, one with $P = E\oplus \overset{k}{\cdots} \oplus E$ and the other with $P = E^*\oplus \overset{k}{\cdots} \oplus E^*$, in connection with which we use the following notation and results (for more details see [8, 15, 19, 30]).

If (q^i, u^ℓ) are local coordinates on P and $\{e_\alpha\}$ is a local basis of sections of E, then a local basis of $\widetilde{\tau}_P : \mathcal{T}^E P \to P$ is given by the family $\{\mathcal{X}_\alpha, \mathcal{V}_\ell\}$ where

$$\mathcal{X}_\alpha(p) = \left(e_\alpha(\pi(p)); \rho^i_\alpha(\pi(p))\frac{\partial}{\partial q^i}\bigg|_p\right) \qquad \text{and} \qquad \mathcal{V}_\ell(p) = \left(0_{\pi(p)}; \frac{\partial}{\partial u^\ell}\bigg|_p\right). \tag{3.7}$$

The Lie bracket of two sections of $\mathcal{T}^E P$ is characterized by the relations

$$[\![\mathcal{X}_\alpha, \mathcal{X}_\beta]\!]^\pi = \mathcal{C}^\gamma_{\alpha\beta}\mathcal{X}_\gamma \quad [\![\mathcal{X}_\alpha, \mathcal{V}_\ell]\!]^\pi = 0 \quad [\![\mathcal{V}_\ell, \mathcal{V}_\varphi]\!]^\pi = 0, \tag{3.8}$$

and the exterior differential is therefore determined by

$$\begin{aligned} &\mathrm{d}^{\mathcal{T}^E P} q^i = \rho^i_\alpha \mathcal{X}^\alpha, && \mathrm{d}^{\mathcal{T}^E P} u^\ell = \mathcal{V}^\ell \\ &\mathrm{d}^{\mathcal{T}^E P} \mathcal{X}^\gamma = -\tfrac{1}{2}\mathcal{C}^\gamma_{\alpha\beta}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta, && \mathrm{d}^{\mathcal{T}^E P} \mathcal{V}^\ell = 0, \end{aligned} \tag{3.9}$$

where $\{\mathcal{X}^\alpha, \mathcal{V}^\ell\}$ is the dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}_\ell\}$.

4. Classical field theories on Lie algebroids: a *k*-symplectic approach

In this section, the k-symplectic formalism for first-order classical field theories (see [14, 36, 41]) is extended to the setting of Lie algebroids. Regarding a Lie algebroid E as a generalization of the tangent bundle of Q, we define the analog of the field solution of the field equations, and we study the analogs of the geometric structures of the standard k-symplectic formalism. Lagrangian and Hamiltonian formalisms are developed in subsections 4.1 and 4.2, respectively, and it is verified that the standard Lagrangian and Hamiltonian k-symplectic formalisms are particular cases of the formalisms developed here. Throughout this section we consider a Lie algebroid $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ on the manifold Q and denote this Lie algebroid itself by E.

4.1. Lagrangian formalism

4.1.1. The manifold $\overset{k}{\oplus} E$*.* The standard k-symplectic Lagrangian formalism is developed on the bundle of k^1-velocities of Q, $T^1_k Q$, that is the Whitney sum of k copies of TQ. Since we are thinking of a Lie algebroid E as a substitute for the tangent bundle, it is natural to consider the Whitney sum of k copies of the Lie algebroid E, which we denote by $\overset{k}{\oplus} E = E\oplus \overset{k}{\cdots} \oplus E$, and the projection map $\widetilde{\tau} : \overset{k}{\oplus} E \to Q$, given by $\widetilde{\tau}\left(e_{1_q}, \ldots, e_{k_q}\right) = q$. If (q^i, y^α) are local coordinates on $\tau^{-1}(U) \subseteq E$, then the induced local coordinates $\left(q^i, y^\alpha_A\right)$ on $\widetilde{\tau}^{-1}(U) \subseteq \overset{k}{\oplus} E$ are given by

$$q^i\left(e_{1_q}, \ldots, e_{k_q}\right) = q^i(q), \qquad y^\alpha_A\left(e_{1_q}, \ldots, e_{k_q}\right) = y^\alpha\left(e_{A_q}\right).$$

Remark 4.1. Consider the standard case in which $E = TQ$, $\rho_{TQ} = id_{TQ}$. If we fix local coordinates (q^i) on Q, then we have the natural basis of $\mathrm{Sec}(TQ) = \mathfrak{X}(Q)$ given by $\{\partial/\partial q^i\}$. For this basis of sections, $\mathcal{C}^\gamma_{\alpha\beta} = 0$ and the set $\mathrm{Sec}\,(\overset{k}{\oplus} TQ) = \mathrm{Sec}\left(T^1_k Q\right)$ is the set $\mathfrak{X}^k(Q)$ of k-vector fields on Q.

4.1.2. The Lagrangian prolongation. Consider the prolongation Lie algebroid E over the fibration $\widetilde{\tau}\colon \overset{k}{\oplus} E \to Q$, that is (see section 3.4),

$$\mathcal{T}^E(\overset{k}{\oplus}E) = \{(e_q, v_{\mathbf{b}_q}) \in E \times T(\overset{k}{\oplus}E)/\rho_E(e_q) = T\widetilde{\tau}(v_{\mathbf{b}_q})\}, \tag{4.1}$$

where $\mathbf{b}_q \in \overset{k}{\oplus} E_q$. The following properties are derived from the general characteristics of prolongation Lie algebroids (see, for instance, [8, 19, 30]):

(i) $\mathcal{T}^E(\overset{k}{\oplus}E) \equiv E \times_{TQ} T(\overset{k}{\oplus}E)$, with projection

$$\widetilde{\tau}_{\overset{k}{\oplus}E} : \mathcal{T}^E(\overset{k}{\oplus}E) \equiv E \times_{TQ} T(\overset{k}{\oplus}E) \longrightarrow \overset{k}{\oplus}E$$

has the Lie algebroid structure $([\![\cdot,\cdot]\!]^{\widetilde{\tau}}, \rho^{\widetilde{\tau}})$, where the anchor map

$$\rho^{\widetilde{\tau}} = E \times_{TQ} T(\overset{k}{\oplus}E) : \mathcal{T}^E(\overset{k}{\oplus}E) \to T(\overset{k}{\oplus}E)$$

is the canonical projection to the second factor. We will refer to this Lie algebroid as the *Lagrangian prolongation*.

(ii) If $\left(q^i, y^\alpha_A\right)$ are local coordinates on $\overset{k}{\oplus} E$, then the induced local coordinates on $\mathcal{T}^E(\overset{k}{\oplus}E) \equiv E \times_{TQ} T(\overset{k}{\oplus}E)$ are

$$\left(q^i, y^\alpha_A, z^\alpha, w^\alpha_A\right)_{1\leqslant i\leqslant n, 1\leqslant A\leqslant k, 1\leqslant\alpha\leqslant m,}$$

where

$$\begin{aligned} q^i\left(e_q, v_{\mathbf{b}_q}\right) &= q^i(q), & y^\alpha_A\left(e_q, v_{\mathbf{b}_q}\right) &= y^\alpha_A(\mathbf{b}_q),\\ z^\alpha\left(e_q, v_{\mathbf{b}_q}\right) &= y^\alpha(e_q), & w^\alpha_A\left(e_q, v_{\mathbf{b}_q}\right) &= v_{\mathbf{b}_q}\left(y^\alpha_A\right). \end{aligned} \tag{4.2}$$

(iii) The set $\{\mathcal{X}_\alpha, \mathcal{V}^A_\alpha\}$ given by

$$\begin{array}{rcccl} \mathcal{X}_\alpha : & \overset{k}{\oplus} E & \to & & \mathcal{T}^E(\overset{k}{\oplus}E) \equiv E \times_{TQ} T(\overset{k}{\oplus}E)\\ & \mathbf{b}_q & \mapsto & \mathcal{X}_\alpha(\mathbf{b}_q) = & \left(e_\alpha(q); \rho^i_\alpha(q)\left.\dfrac{\partial}{\partial q^i}\right|_{\mathbf{b}_q}\right)\\ \mathcal{V}^A_\alpha : & \overset{k}{\oplus}E & \to & & \mathcal{T}^E(\overset{k}{\oplus}E) \equiv E \times_{TQ} T(\overset{k}{\oplus}E)\\ & \mathbf{b}_q & \mapsto & \mathcal{V}^A_\alpha(\mathbf{b}_q) = & \left(0_q; \left.\dfrac{\partial}{\partial y^\alpha_A}\right|_{\mathbf{b}_q}\right) \end{array} \tag{4.3}$$

is a local basis of $\mathrm{Sec}(\mathcal{T}^E(\overset{k}{\oplus}E))$, the set of sections of $\widetilde{\tau}_{\overset{k}{\oplus}E}$ (see (3.7)).

(iv) The anchor map $\rho^{\widetilde{\tau}} : \mathcal{T}^E(\overset{k}{\oplus}E) \to T(\overset{k}{\oplus}E)$ allows us to associate a vector field with each section $\xi\colon \overset{k}{\oplus}E \to \mathcal{T}^E(\overset{k}{\oplus}E) \equiv E \times_{TQ} T(\overset{k}{\oplus}E)$ of $\widetilde{\tau}_{\overset{k}{\oplus}E}$. Locally, if

$$\xi = \xi^\alpha \mathcal{X}_\alpha + \xi^\alpha_A \mathcal{V}^A_\alpha \in \mathrm{Sec}(\mathcal{T}^E(\overset{k}{\oplus}E)),$$

then the associated vector field is given by

$$\rho^{\widetilde{\tau}}(\xi) = \rho^i_\alpha \xi^\alpha \frac{\partial}{\partial q^i} + \xi^\alpha_A \frac{\partial}{\partial y^\alpha_A} \in \mathfrak{X}(\overset{k}{\oplus}E). \tag{4.4}$$

(v) The Lie bracket of two sections of $\widetilde{\tau}_{\overset{k}{\oplus}E}$ is characterized by (see (3.8))

$$[\![\mathcal{X}_\alpha, \mathcal{X}_\beta]\!]^{\widetilde{\tau}} = \mathcal{C}^\gamma_{\alpha\beta}\mathcal{X}_\gamma \quad \left[\!\left[\mathcal{X}_\alpha, \mathcal{V}^B_\beta\right]\!\right]^{\widetilde{\tau}} = 0 \quad \left[\!\left[\mathcal{V}^A_\alpha, \mathcal{V}^B_\beta\right]\!\right]^{\widetilde{\tau}} = 0. \tag{4.5}$$

(vi) If $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha_A\}$ is the dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}^A_\alpha\}$, then the exterior differential is given locally (see (3.9)) by

$$\begin{aligned} &\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus}E)} f = \rho^i_\alpha \frac{\partial f}{\partial q^i}\mathcal{X}^\alpha + \frac{\partial f}{\partial y^\alpha_A}\mathcal{V}^\alpha_A, && \text{for all} \quad f \in \mathcal{C}^\infty(\overset{k}{\oplus}E) \\ &\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus}E)} \mathcal{X}^\gamma = -\frac{1}{2}\mathcal{C}^\gamma_{\alpha\beta}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta, && \mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus}E)}\mathcal{V}^\gamma_A = 0. \end{aligned} \tag{4.6}$$

Remark 4.2. In the particular case $E = TQ$, the manifold $\mathcal{T}^E(\overset{k}{\oplus}E)$ reduces to $T(T^1_kQ)$:

$$\begin{aligned} \mathcal{T}^{TQ}\big(\overset{k}{\oplus}TQ\big) &= \mathcal{T}^{TQ}\big(T^1_kQ\big) \\ &= \big\{(u_q, v_{\mathbf{w}_q}) \in TQ \times T\big(T^1_kQ\big)/u_q = T\big(\tau^k_Q\big)(v_{\mathbf{w}_q})\big\} \\ &= \big\{\big(T\big(\tau^k_Q\big)(v_{\mathbf{w}_q}), v_{\mathbf{w}_q}\big) \in TQ \times T\big(T^1_kQ\big)/\mathbf{w}_q \in T^1_kQ\big\} \\ &\equiv \big\{v_{\mathbf{w}_q} \in T\big(T^1_kQ\big)/\mathbf{w}_q \in T^1_kQ\big\} \equiv T\big(T^1_kQ\big). \end{aligned} \tag{4.7}$$

4.1.3. Liouville sections and vertical endomorphisms. On $\mathcal{T}^E(\overset{k}{\oplus}E)$ we define two families of canonical objects, *Liouville sections* and *vertical endomorphisms* which correspond to the *Liouville vector fields* and *k-tangent structure* on T^1_kQ (see [14, 36, 41]).

Vertical A-lifting (see for instance [8]**).** An element $(e_q, v_{\mathbf{b}_q})$ of $\mathcal{T}^E(\overset{k}{\oplus}E) \equiv E \times_{TQ} T(\overset{k}{\oplus}E)$ is said to be *vertical* if

$$\widetilde{\tau}_1\big(e_q, v_{\mathbf{b}_q}\big) = 0_q \in E, \tag{4.8}$$

where

$$\begin{aligned} \widetilde{\tau}_1 : \quad \mathcal{T}^E(\overset{k}{\oplus}E) \equiv E \times_{TQ} T(\overset{k}{\oplus}E) &\to E, \\ \big(e_q, v_{\mathbf{b}_q}\big) &\mapsto \widetilde{\tau}_1\big(e_q, v_{\mathbf{b}_q}\big) = e_q \end{aligned}$$

is the projection on the first factor E. The vertical elements of $\mathcal{T}^E(\overset{k}{\oplus}E)$ are thus of the form

$$\big(0_q, v_{\mathbf{b}_q}\big) \in \mathcal{T}^E(\overset{k}{\oplus}E) \equiv E \times_{TQ} T(\overset{k}{\oplus}E),$$

where $v_{\mathbf{b}_q} \in T(\overset{k}{\oplus}E)$ and $\mathbf{b}_q \in \overset{k}{\oplus}E$. In particular, the tangent vector $v_{\mathbf{b}_q}$ is $\widetilde{\tau}$-vertical, since by (4.1)

$$0_q = T_{\mathbf{b}_q}\widetilde{\tau}\big(v_{\mathbf{b}_q}\big).$$

In a local coordinate system $\big(q^i, y^\alpha_A\big)$ on $\overset{k}{\oplus}E$, if $\big(e_q, v_{\mathbf{b}_q}\big) \in \mathcal{T}^E(\overset{k}{\oplus}E)$ is vertical, then $e_q = 0_q$ and

$$v_{\mathbf{b}_q} = u^\alpha_A \frac{\partial}{\partial y^\alpha_A}\bigg|_{\mathbf{b}_q} \in T_{\mathbf{b}_q}(\overset{k}{\oplus}E).$$

Definition 4.3. *For each $A = 1, \ldots, k$, the vertical A-lifting is defined as the mapping*

$$\begin{aligned} \xi^{V_A} : E \times_Q (\overset{k}{\oplus}E) &\longrightarrow \mathcal{T}^E(\overset{k}{\oplus}E) \equiv E \times_{TQ} T(\overset{k}{\oplus}E) \\ (e_q, \mathbf{b}_q) &\longmapsto \xi^{V_A}(e_q, \mathbf{b}_q) = \big(0_q, (e_q)^{V_A}_{\mathbf{b}_q}\big), \end{aligned} \tag{4.9}$$

where $e_q \in E$, $\mathbf{b}_q = (b_{1q}, \ldots, b_{kq}) \in \overset{k}{\oplus} E$ *and the vector* $(e_q)^{V_A}_{\mathbf{b}_q} \in T_{\mathbf{b}_q}(\overset{k}{\oplus} E)$ *is given by*

$$(e_q)^{V_A}_{\mathbf{b}_q} f = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} f(b_{1_q}, \ldots, b_{A_q} + s e_q, \ldots, b_{k_q}), \qquad 1 \leqslant A \leqslant k, \tag{4.10}$$

for an arbitrary function $f \in \mathcal{C}^\infty(\overset{k}{\oplus} E)$.

The local expression of $(e_q)^{V_A}_{\mathbf{b}_q}$ is

$$(e_q)^{V_A}_{\mathbf{b}_q} = y^\alpha(e_q) \frac{\partial}{\partial y^\alpha_A}\bigg|_{\mathbf{b}_q} \in T_{\mathbf{b}_q}(\overset{k}{\oplus} E), \qquad 1 \leqslant A \leqslant k. \tag{4.11}$$

Since $(e_q)^{V_A}_{\mathbf{b}_q} \in T_{\mathbf{b}_q}(\overset{k}{\oplus} E)$ is $\widetilde{\tau}$-vertical, $\xi^{V_A}(e_q, \mathbf{b}_q)$ is a vertical element of $\mathcal{T}^E(\overset{k}{\oplus} E)$, and by (4.3), (4.9) and (4.11) its local expression is

$$\xi^{V_A}(e_q, \mathbf{b}_q) = \left(0_q, y^\alpha(e_q) \frac{\partial}{\partial y^\alpha_A}\bigg|_{\mathbf{b}_q}\right) = y^\alpha(e_q) \mathcal{V}^A_\alpha(\mathbf{b}_q), \qquad 1 \leqslant A \leqslant k. \tag{4.12}$$

Remark 4.4.

(i) In the standard case ($E = TQ$, $\rho_{TQ} = id_{TQ}$), given $e_q \in T_q Q$ and $\mathbf{v}_q = (v_{1q}, \ldots, v_{kq}) \in T^1_k Q$,

$$(e_q)^{V_A}_{\mathbf{v}_q}(f) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} f(v_{1_q}, \ldots, v_{A_q} + s e_q, \ldots, v_{k_q}), \qquad 1 \leqslant A \leqslant k,$$

that is the vertical A-lift to $T^1_k Q$ of the tangent vector e_q (see, for instance, [14, 36, 41]).

(ii) When $k = 1$, $\xi^{V_1} \equiv \xi^V : E \times_Q E \to \mathcal{T}^E E$ is the *vertical lifting map* introduced by Martínez in [30].

Liouville sections. The Ath *Liouville section* $\widetilde{\Delta}_A$ is the section of $\widetilde{\tau}_{\overset{k}{\oplus} E} : \mathcal{T}^E(\overset{k}{\oplus} E) \to \overset{k}{\oplus} E$ given by

$$\begin{array}{rcl} \widetilde{\Delta}_A : \overset{k}{\oplus} E & \to & \mathcal{T}^E(\overset{k}{\oplus} E) \equiv E \times_{TQ} T(\overset{k}{\oplus} E) \\ \mathbf{b}_q & \mapsto & \widetilde{\Delta}_A(\mathbf{b}_q) = \xi^{V_A}(pr_A(\mathbf{b}_q), \mathbf{b}_q) = \xi^{V_A}(b_{Aq}, \mathbf{b}_q), \end{array} \qquad 1 \leqslant A \leqslant k,$$

where $\mathbf{b}_q = (b_{1q}, \ldots, b_{kq}) \in \overset{k}{\oplus} E$ and $pr_A : \overset{k}{\oplus} E \to E$ is the canonical projection over the Ath copy of E in $\overset{k}{\oplus} E$. From the local expression (4.12) of ξ^{V_A}, and since

$$y^\alpha(b_{Aq}) = y^\alpha_A(b_{1q}, \ldots, b_{kq}) = y^\alpha_A(\mathbf{b}_q),$$

$\widetilde{\Delta}_A$ has the local expression

$$\widetilde{\Delta}_A = \sum_{\alpha=1}^m y^\alpha_A \mathcal{V}^A_\alpha, \qquad 1 \leqslant A \leqslant k. \tag{4.13}$$

Remark 4.5. In the standard case, $\widetilde{\Delta}_A$ is the vector field:

$$\begin{array}{rcl} \Delta_A : \quad T^1_k Q & \to & T(T^1_k Q) \\ \mathbf{v}_q = (v_{1q}, \ldots, (v_k)_q) & \mapsto & (v_{Aq})^{V_A}_{\mathbf{v}_q}, \end{array}$$

that is the Ath canonical vector field on $T^1_k Q$.

In the standard Lagrangian k-symplectic formalism, the canonical vector fields $\Delta_1, \ldots, \Delta_k$ allow us to define the energy function. Analogously as we will see below, the energy function can be defined in the Lie algebroid setting using the Liouville sections $\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_k$.

Vertical endomorphisms on $\mathcal{T}^E(\overset{k}{\oplus}E)$. The second important family of canonical geometric elements on $\mathcal{T}^E(\overset{k}{\oplus}E)$ is the family of vertical endomorphisms $\widetilde{J}^1, \ldots, \widetilde{J}^k$.

Definition 4.6. *For $A = 1, \ldots, k$ the Ath vertical endomorphism on $\mathcal{T}^E(\overset{k}{\oplus} E) \equiv E \times_{TQ} T(\overset{k}{\oplus}E)$ is the mapping*

$$\begin{array}{rccl} \widetilde{J}^A: & \mathcal{T}^E(\overset{k}{\oplus} E) & \to & \mathcal{T}^E(\overset{k}{\oplus} E) \\ & (e_q, v_{\mathbf{b}_q}) & \mapsto & \widetilde{J}^A(e_q, v_{\mathbf{b}_q}) = \xi^{V_A}(e_q, \mathbf{b}_q), \end{array} \tag{4.14}$$

where $e_q \in E$, $\mathbf{b}_q = (b_{1q}, \ldots, b_{kq}) \in \overset{k}{\oplus}E$ and $v_{\mathbf{b}_q} \in T_{\mathbf{b}_q}(\overset{k}{\oplus}E)$.

Lemma 4.7. *Let $\{\mathcal{X}_\alpha, \mathcal{V}^A_\alpha\}$ be a local basis of* $\mathrm{Sec}(\mathcal{T}^E(\overset{k}{\oplus}E))$ *and let $\{\mathcal{X}^\alpha, \mathcal{V}_A^\alpha\}$ be its dual basis. The corresponding local expression of $\widetilde{J}^A$ is*

$$\widetilde{J}^A = \sum_{\alpha=1}^{m} \mathcal{V}^A_\alpha \otimes \mathcal{X}^\alpha, \qquad 1 \leqslant A \leqslant k. \tag{4.15}$$

Proof. By (4.3) and (4.12),

$$\widetilde{J}^A(\mathcal{X}_\alpha(\mathbf{b}_q)) = \xi^{V_A}(e_\alpha(q), \mathbf{b}_q) = y^\beta(e_\alpha(q))\mathcal{V}^A_\beta(\mathbf{b}_q) = \mathcal{V}^A_\alpha(\mathbf{b}_q),$$
$$\widetilde{J}^A(\mathcal{V}^B_\alpha(\mathbf{b}_q)) = \xi^{V_A}(0_q, \mathbf{b}_q) = 0_{\mathbf{b}_q},$$

for each $A, B = 1, \ldots, k$, $\alpha = 1 \ldots, m$, where $\mathbf{b}_q \in \overset{k}{\oplus}E$ is an arbitrary element of $\overset{k}{\oplus}E$. □

Remark 4.8.

(i) In the standard case ($E = TQ$, $\rho = id_{TQ}$), the $\widetilde{J}^A$ constitute the canonical k-tangent structure $J^1, \ldots, J^k$ on T^1_kQ.
(ii) When $k = 1$, $\widetilde{J}$ is the vertical endomorphism defined by Martínez [31] on $\mathcal{T}^E(TQ)$, the prolongation of the Lie algebroid E over $\tau_Q : TQ \to Q$.

4.1.4. Second-order partial differential equations (SOPDES). In the standard k-symplectic Lagrangian formalism one obtains the solutions of the Euler–Lagrange equations as integral sections of certain second-order partial differential equations on T^1_kQ. In order to introduce the analogous object on Lie algebroids, we note that in the standard case a SOPDE ξ is a section of the maps

$$\begin{array}{rccl} \tau^k_{T^1_kQ}: & T^1_k(T^1_kQ) & \to & T^1_kQ \\ & (v_{1\mathbf{w}_q}, \ldots, v_{k\mathbf{w}_q}) & \mapsto & \mathbf{w}_q \end{array}$$

and

$$\begin{array}{rccl} T^1_k(\tau^k_Q): & T^1_k(T^1_kQ) & \to & T^1_kQ \\ & (v_{1\mathbf{w}_q}, \ldots, v_{k\mathbf{w}_q}) & \mapsto & (T_{\mathbf{w}_q}(\tau^k_Q)(v_{1\mathbf{w}_q}), \ldots, T_{\mathbf{w}_q}(\tau^k_Q)(v_{k\mathbf{w}_q})), \end{array}$$

where $\tau_Q^k : T_k^1 Q \to Q$ denotes the canonical projection of the tangent bundle of k^1-velocities. Since $T_k^1\big(T_k^1 Q\big)$ is the Whitney sum of k copies of $T\big(T_k^1 Q\big)$, it is natural to think that in the Lie algebroid context its role will be played by the Whitney sum of k copies of $\mathcal{T}^E(\overset{k}{\oplus}E)$, that is

$$(\mathcal{T}^E)^1_k(\overset{k}{\oplus}E) := \mathcal{T}^E(\overset{k}{\oplus}E)\oplus \overset{k}{\cdots} \oplus\mathcal{T}^E(\overset{k}{\oplus}E).$$

Furthermore, the maps

$$\begin{array}{rcl} \widetilde{\tau}^k_{\overset{k}{\oplus}E} : \quad (\mathcal{T}^E)^1_k(\overset{k}{\oplus}E) \equiv \mathcal{T}^E(\overset{k}{\oplus}E)\oplus \overset{k}{\cdots} \oplus\mathcal{T}^E(\overset{k}{\oplus}E) & \to & \overset{k}{\oplus}E \\ \big(\big(a_{1q}, v_{1\mathbf{b}_q}\big), \ldots, \big(a_{kq}, v_{k\mathbf{b}_q}\big)\big) & \mapsto & \mathbf{b}_q \end{array}$$

and

$$\begin{array}{rcl} \widetilde{\tau}^k_1 : \quad (\mathcal{T}^E)^1_k(\overset{k}{\oplus}E) \equiv \mathcal{T}^E(\overset{k}{\oplus}E)\oplus \overset{k}{\cdots} \oplus\mathcal{T}^E(\overset{k}{\oplus}E) & \to & \overset{k}{\oplus}\, E \\ \big(\big(a_{1q}, v_{1\mathbf{b}_q}\big), \ldots, \big(a_{kq}, v_{k\mathbf{b}_q}\big)\big) & \mapsto & \big(a_{1q}, \ldots, a_{kq}\big), \end{array}$$

play the roles of $\tau^k_{T_k^1 Q}$ and $T_k^1\big(\tau_Q^k\big)$, respectively. In fact, when $E = TQ$ there is a diffeomorphism between $T\big(T_k^1 Q\big)$ and $\mathcal{T}^{TQ}\big(T_k^1 Q\big)$ given by (see remark 4.2)

$$\begin{array}{l} T\big(T_k^1 Q\big) \equiv \mathcal{T}^{TQ}\big(T_k^1 Q\big) = (TQ) \times_{TQ} T\big(T_k^1 Q\big) \equiv T\big(T_k^1 Q\big) \\ v_{\mathbf{w}_q} \equiv \big(T_{\mathbf{w}_q}\big(\tau_Q^k\big)\big(v_{\mathbf{w}_q}\big), v_{\mathbf{w}_q}\big), \end{array}$$

under which diffeomorphism the map

$$\widetilde{\tau}^k_{\overset{k}{\oplus}TQ} : (\mathcal{T}^{TQ})^1_k\big(T_k^1 Q\big) \equiv T_k^1\big(T_k^1 Q\big) \to T_k^1 Q$$

corresponds to $\tau^k_{T_k^1 Q} : T_k^1\big(T_k^1 Q\big) \to T_k^1 Q$, since

$$\begin{aligned} \widetilde{\tau}^k_{\overset{k}{\oplus}TQ}\big(\big(T_{\mathbf{w}_q}\big(\tau_Q^k\big)\big(v_{1\mathbf{w}_q}\big), v_{1\mathbf{w}_q}\big), \ldots, \big(T_{\mathbf{w}_q}\big(\tau_Q^k\big)\big(v_{k\mathbf{w}_q}\big), v_{k\mathbf{w}_q}\big)\big) &= \mathbf{w}_q \\ &= \tau^k_{T_k^1 Q}\big(v_{1\mathbf{w}_q}, \ldots, v_{k\mathbf{w}_q}\big). \end{aligned}$$

and the map

$$\widetilde{\tau}^k_1 : (\mathcal{T}^{TQ})^1_k\big(T_k^1 Q\big) \equiv T_k^1\big(T_k^1 Q\big) \to T_k^1 Q$$

corresponds to $T_k^1\big(\tau_Q^k\big) : T_k^1\big(T_k^1 Q\big) \to T_k^1 Q$, since

$$\begin{aligned} &\widetilde{\tau}^k_1\big(\big(T_{\mathbf{w}_q}\big(\tau_Q^k\big)\big(v_{1\mathbf{w}_q}\big), v_{1\mathbf{w}_q}\big), \ldots, \big(T_{\mathbf{w}_q}\big(\tau_Q^k\big)\big(v_{k\mathbf{w}_q}\big), v_{k\mathbf{w}_q}\big)\big) \\ &\quad = \big(T_{\mathbf{w}_q}\big(\tau_Q^k\big)\big(v_{1\mathbf{w}_q}\big), \ldots, T_{\mathbf{w}_q}\big(\tau_Q^k\big)\big(v_{k\mathbf{w}_q}\big)\big) = T_k^1\big(\tau_Q^k\big)\big(v_{1\mathbf{w}_q}, \ldots, v_{k\mathbf{w}_q}\big). \end{aligned}$$

Remark 4.9. For simplicity we denote by $\big(\mathbf{a}_q, \mathbf{v}_{\mathbf{b}_q}\big)$ an element

$$\big(\big(a_{1q}, v_{1\mathbf{b}_q}\big), \ldots, \big(a_{kq}, v_{k\mathbf{b}_q}\big)\big)$$

of $(\mathcal{T}^E)^1_k(\overset{k}{\oplus}\, E) \equiv \mathcal{T}^E(\overset{k}{\oplus}\, E)\oplus \overset{k}{\cdots} \oplus\mathcal{T}^E(\overset{k}{\oplus}\, E)$, where $\mathbf{a}_q : = (a_{1q}, \ldots, a_{kq}) \in\overset{k}{\oplus}\, E$ and $\mathbf{v}_{\mathbf{b}_q} : = \big(v_{1\mathbf{b}_q}, \ldots, v_{k\mathbf{b}_q}\big) \in T_k^1(\overset{k}{\oplus}E)$.

We are now in a position to introduce SOPDEs on Lie algebroids.

Definition 4.10. *A second-order partial differential equation (*SOPDE*) on* $\overset{k}{\oplus}\, E$ *is a map* $\xi\colon \overset{k}{\oplus}E \to (\mathcal{T}^E)^1_k(\overset{k}{\oplus}E)$ *that is a section of* $\widetilde{\tau}^k_{\overset{k}{\oplus}E}$ *and* $\widetilde{\tau}^k_1$.

Since $(\mathcal{T}^E)^1_k(\overset{k}{\oplus}E)$ is the Whitney sum of k copies of $\mathcal{T}^E(\overset{k}{\oplus}E)$, we deduce that to give a section ξ of $\widetilde{\tau}^k_{\overset{k}{\oplus}E}$ is equivalent to giving a family of k sections $\xi_1, \ldots, \xi_k$, of the Lagrangian prolongation $\mathcal{T}^E(\overset{k}{\oplus}E)$ obtained by projecting ξ on each factor.

To characterize SOPDES on Lie algebroids we need the following.

Definition 4.11. *The set*

$$Adm(E) = \left\{\left(\mathbf{a}_q, \mathbf{v}_{\mathbf{b}_q}\right) \in (\mathcal{T}^E)^1_k(\overset{k}{\oplus}\, E) \mid \widetilde{\tau}^k_1\left(\mathbf{a}_q, \mathbf{v}_{\mathbf{b}_q}\right) = \widetilde{\tau}^k_{\overset{k}{\oplus}E}\left(\mathbf{a}_q, \mathbf{v}_{\mathbf{b}_q}\right)\right\}$$
$$= \left\{\left(\mathbf{a}_q, \mathbf{v}_{\mathbf{b}_q}\right) \in (\mathcal{T}^E)^1_k(\overset{k}{\oplus}\, E) \mid \mathbf{a}_q = \mathbf{b}_q\right\} \tag{4.16}$$

is called the set of admissible points of E.

Proposition 4.12. *Let $\xi = (\xi_1, \ldots, \xi_k) : \overset{k}{\oplus}\, E \to (\mathcal{T}^E)^1_k(\overset{k}{\oplus}\, E)$ be a section of $\widetilde{\tau}^k_{\overset{k}{\oplus}E}$. The following statements are equivalent.*

(i) ξ takes values in $Adm(E)$.
(ii) ξ is a SOPDE, that is $\widetilde{\tau}^k_1 \circ \xi = \mathrm{id}_{\overset{k}{\oplus}E}$.
(iii) $\widetilde{J}^A(\xi_A) = \widetilde{\Delta}_A$ for all $A = 1, \ldots, k$.

Proof. From (4.16) it is easy to prove that (i) and (ii) are equivalent. The equivalence of (i) and (iii) is a direct consequence of the definitions of $\widetilde{J}^A$, $\widetilde{\Delta}_A$ and ξ^{V_A}. □

Using proposition 4.12 (iii), one easily deduces that the local expression of a SOPDE $\xi = (\xi_1, \ldots, \xi_k)$ is

$$\xi_A = y^\alpha_A \mathcal{X}_\alpha + (\xi_A)^\alpha_B \mathcal{V}^B_\alpha,$$

where $(\xi_A)^\alpha_B \in \mathcal{C}^\infty(\overset{k}{\oplus}E)$.

Proposition 4.13. *Let $\xi = (\xi_1, \ldots, \xi_k) : \overset{k}{\oplus}\, E \to (\mathcal{T}^E)^1_k(\overset{k}{\oplus}\, E)$ be a section of $\widetilde{\tau}^k_{\overset{k}{\oplus}E}$. Then*

$$(\rho^{\widetilde{\tau}}(\xi_1), \ldots, \rho^{\widetilde{\tau}}(\xi_k)) \colon \overset{k}{\oplus}E \to T^1_k(\overset{k}{\oplus}E)$$

is a k-vector field on $\overset{k}{\oplus}E$, where

$$\rho^{\widetilde{\tau}} : \mathcal{T}^E(\overset{k}{\oplus}E) \equiv E \times_{TQ} T(\overset{k}{\oplus}E) \to T(\overset{k}{\oplus}E)$$

is the anchor map of the Lie algebroid $\mathcal{T}^E(\overset{k}{\oplus}E)$.

Proof. Directly by section 4.1.2 (vi). □

In local coordinates

$$\rho^{\widetilde{\tau}}(\xi_A) = \rho^i_\alpha y^\alpha_A \frac{\partial}{\partial q^i} + (\xi_A)^\alpha_B \frac{\partial}{\partial y^\alpha_B} \in \mathfrak{X}(\overset{k}{\oplus}E). \tag{4.17}$$

Definition 4.14. *A map*

$$\eta : \mathbb{R}^k \to \overset{k}{\oplus}\, E$$

is an integral section of the SOPDE ξ *if* η *is an integral section of the k-vector field* $(\rho^{\widetilde{\tau}}(\xi_1), \ldots, \rho^{\widetilde{\tau}}(\xi_k))$ *associated with* ξ*, that is*

$$(\rho^{\widetilde{\tau}}(\xi_A))(\eta(\mathbf{t})) = \eta_*(\mathbf{t})\left(\frac{\partial}{\partial t^A}\bigg|_{\mathbf{t}}\right), \qquad 1 \leqslant A \leqslant k. \tag{4.18}$$

If η is written locally as $\eta(\mathbf{t}) = (\eta^i(\mathbf{t}), \eta^\alpha_A(\mathbf{t}))$, then from (4.17) we deduce that (4.18) is locally equivalent to the identities

$$\frac{\partial \eta^i}{\partial t^A}\bigg|_{\mathbf{t}} = \eta^\alpha_A(\mathbf{t})\rho^i_\alpha(\widetilde{\tau}(\eta(\mathbf{t})), \qquad \frac{\partial \eta^\beta_B}{\partial t^A}\bigg|_{\mathbf{t}} = (\xi_A)^\beta_B(\eta(\mathbf{t})), \tag{4.19}$$

where $\widetilde{\tau} : \overset{k}{\oplus} E \to Q$ is the canonical projection.

4.1.5. Lagrangian formalism. In this section we develop an intrinsic and global geometric framework that allows us to write the Euler–Lagrange equations associated with a Lagrangian function $L : \overset{k}{\oplus} E \to \mathbb{R}$ on a Lie algebroid. We first introduce some geometric elements associated with L.

Poincaré–Cartan sections. The *Poincaré–Cartan 1-sections* Θ^A_L are defined by

$$\begin{array}{rcl} \Theta^A_L : \overset{k}{\oplus} E & \longrightarrow & (\mathcal{T}^E(\overset{k}{\oplus} E))^* \\ \mathbf{b}_q & \longmapsto & \Theta^A_L(\mathbf{b}_q), \end{array}$$

where

$$\begin{array}{rcl} \Theta^A_L(\mathbf{b}_q) : \quad (\mathcal{T}^E(\overset{k}{\oplus} E))_{\mathbf{b}_q} & \longrightarrow & \mathbb{R} \\ Z_{\mathbf{b}_q} = (e_q, v_{\mathbf{b}_q}) & \longmapsto & (\Theta^A_L)_{\mathbf{b}_q}(Z_{\mathbf{b}_q}) = (\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E)} L)_{\mathbf{b}_q}((\widetilde{J}^A)_{\mathbf{b}_q}(Z_{\mathbf{b}_q})). \end{array}$$

Using (4.6) with $f = L$,

$$(\Theta^A_L)(\mathbf{b}_q)Z_{\mathbf{b}_q} = (\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E)} L)_{\mathbf{b}_q}((\widetilde{J}^A)_{\mathbf{b}_q} Z_{\mathbf{b}_q}) = (\rho^{\widetilde{\tau}}((\widetilde{J}^A)_{\mathbf{b}_q} Z_{\mathbf{b}_q}))L, \tag{4.20}$$

where $\mathbf{b}_q \in \overset{k}{\oplus} E$, $Z_{\mathbf{b}_q} \in [\mathcal{T}^E(\overset{k}{\oplus} E)]_{\mathbf{b}_q}$ and $\rho^{\widetilde{\tau}}((\widetilde{J}^A)_{\mathbf{b}_q} Z_{\mathbf{b}_q}) \in T_{\mathbf{b}_q}(\overset{k}{\oplus} E)$.

The Poincaré–Cartan 2-sections

$$\Omega^A_L : \overset{k}{\oplus} E \to (\mathcal{T}^E(\overset{k}{\oplus} E))^* \wedge (\mathcal{T}^E(\overset{k}{\oplus} E))^*, \qquad 1 \leqslant A \leqslant k$$

are defined by

$$\Omega^A_L := -\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E)}\Theta^A_L, \qquad 1 \leqslant A \leqslant k,$$

that is

$$\begin{aligned} \Omega^A_L(\xi_1, \xi_2) &= -\mathrm{d}\Theta^A_L(\xi_1, \xi_2) \\ &= [\rho^{\widetilde{\tau}}(\xi_2)](\Theta^A_L(\xi_1)) - [\rho^{\widetilde{\tau}}(\xi_1)](\Theta^A_L(\xi_2)) + \Theta^A_L([\![\xi_1, \xi_2]\!]^{\widetilde{\tau}}), \end{aligned} \tag{4.21}$$

where $\xi_1, \xi_2 \in \mathrm{Sec}(\mathcal{T}^E(\overset{k}{\oplus} E))$ and $(\rho^{\widetilde{\tau}}, [\![\cdot,\cdot]\!]^{\widetilde{\tau}})$ denotes the Lie algebroid structure of $\mathcal{T}^E(\overset{k}{\oplus} E)$ defined in section 4.1.2.

To find the local expressions of Θ^A_L and Ω^A_L, consider $\{\mathcal{X}_\alpha, \mathcal{V}^B_\alpha\}$, a local basis of sections of $\mathrm{Sec}(\mathcal{T}^E(\overset{k}{\oplus} E))$, and its dual basis $\{\mathcal{X}^\alpha, \mathcal{V}^\alpha_B\}$. By (4.4), (4.15) and (4.20),

$$\Theta^A_L = \frac{\partial L}{\partial y^\alpha_A}\mathcal{X}^\alpha, \qquad 1 \leqslant A \leqslant k\,; \tag{4.22}$$

and by the local expressions (4.3), (4.4), (4.5), (4.21) and (4.22),

$$\Omega_L^A = \frac{1}{2}\left(\rho^i_\beta \frac{\partial^2 L}{\partial q^i \partial y^\alpha_A} - \rho^i_\alpha \frac{\partial^2 L}{\partial q^i \partial y^\beta_A} + \mathcal{C}^\gamma_{\alpha\beta}\frac{\partial L}{\partial y^\gamma_A}\right)\mathcal{X}^\alpha \wedge \mathcal{X}^\beta + \frac{\partial^2 L}{\partial y^\beta_B \partial y^\alpha_A}\mathcal{X}^\alpha \wedge \mathcal{V}^\beta_B. \tag{4.23}$$

Remark 4.15.

(i) When $k = 1$, Θ^1_L and Ω^1_L are the Poincaré–Cartan forms of Lagrangian mechanics on Lie algebroids (see, for instance, [8, 31]).

(ii) When $E = TQ$ and $\rho_{TQ} = id_{TQ}$,

$$\Omega^A_L(X, Y) = \omega^A_L(X, Y), \qquad 1 \leqslant A \leqslant k,$$

where X and Y are vector fields on $T^1_k Q$ and $\omega^1_L, \ldots, \omega^k_L$ are the Lagrangian 2-forms of the standard k-symplectic formalism, defined by $\omega^A_L = -\mathrm{d}(\mathrm{d}L \circ J^A)$, where d is the usual exterior derivative.

The energy function. The *energy function* $E_L : \overset{k}{\oplus} E \to \mathbb{R}$ defined by the Lagrangian L is

$$E_L = \sum_{A=1}^k \rho^{\widetilde{\tau}}(\Delta_A)L - L,$$

and from (4.4) and (4.13) one deduces that E_L is given locally by

$$E_L = \sum_{A=1}^k y^\alpha_A \frac{\partial L}{\partial y^\alpha_A} - L. \tag{4.24}$$

Morphisms. We generalize the Euler–Lagrange equations and their solutions to Lie algebroids in terms of a particular Lie algebroid morphism.

In the standard Lagrangian k-symplectic formalism, a solution of the Euler–Lagrange equations is a field $\phi : \mathbb{R}^k \to Q$ with a first prolongation $\phi^{(1)} : \mathbb{R}^k \to T^1_k Q$ that satisfies those equations, that is

$$\sum_{A=1}^k \frac{\partial}{\partial t^A}\bigg|_{\mathbf{t}} \left(\frac{\partial L}{\partial v^i_A}\bigg|_{\phi^{(1)}(\mathbf{t})}\right) = \frac{\partial L}{\partial q^i}\bigg|_{\phi^{(1)}(\mathbf{t})}.$$

The map ϕ naturally induces the Lie algebroid morphism

$$\begin{array}{ccc} T\mathbb{R}^k & \xrightarrow{T\phi} & TQ \\ \downarrow{\scriptstyle \tau_{\mathbb{R}^k}} & & \downarrow{\scriptstyle \tau_Q} \\ \mathbb{R}^k & \xrightarrow[\phi]{} & Q \end{array}$$

and in terms of the canonical basis of sections of $\tau_{\mathbb{R}^k}$, $\left\{\frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^k}\right\}$, the first prolongation of ϕ, $\phi^{(1)}$, can be written as

$$\phi^{(1)}(\mathbf{t}) = \left(T_{\mathbf{t}}\phi\left(\frac{\partial}{\partial t^1}\bigg|_{\mathbf{t}}\right), \ldots, T_{\mathbf{t}}\phi\left(\frac{\partial}{\partial t^k}\bigg|_{\mathbf{t}}\right)\right).$$

For a general Lie algebroid we shall derive the field-theoretic Euler–Lagrange equations in such way that their solutions are Lie algebroid morphisms $\Phi = (\overline{\Phi}, \underline{\Phi})$ between $T\mathbb{R}^k$ and E,

$$\begin{array}{ccc} T\mathbb{R}^k & \xrightarrow{\overline{\Phi}} & E \\ \downarrow{\scriptstyle \tau_{\mathbb{R}^k}} & & \downarrow{\scriptstyle \tau} \\ \mathbb{R}^k & \xrightarrow[\underline{\Phi}]{} & Q \end{array}$$

with an associated map $\widetilde{\Phi} : \mathbb{R}^k \to \overset{k}{\oplus} E$ that satisfies those equations and is given by

$$\begin{array}{rcl} \widetilde{\Phi} : \mathbb{R}^k & \to & \overset{k}{\oplus} E \equiv E \oplus \overset{k}{\cdots} \oplus E \\ \mathbf{t} & \to & (\overline{\Phi}(e_1(\mathbf{t})), \ldots, \overline{\Phi}(e_k(\mathbf{t}))), \end{array}$$

where $\{e_A\}_{A=1}^k$ is a local basis of local sections of $T\mathbb{R}^k$.

If (t^A) and (q^i) are local coordinate systems on $\mathbb{R}^k$ and Q, respectively; $\{e_A\}$ and $\{e_\alpha\}$ local bases of sections of $\tau_{\mathbb{R}^k}$ and E, respectively; and $\{e^A\}$ and $\{e^\alpha\}$ the respective dual bases, then $\underline{\Phi}(\mathbf{t}) = (\phi^i(\mathbf{t}))$ and $\Phi^* e^\alpha = \phi^\alpha_A e^A$ for certain local functions ϕ^i and ϕ^α_A on $\mathbb{R}^k$, the associated map $\widetilde{\Phi}$ is given locally by $\widetilde{\Phi}(\mathbf{t}) = (\phi^i(\mathbf{t}), \phi^\alpha_A(\mathbf{t}))$, and the Lie algebroid morphism conditions (3.6) are

$$\rho^i_\alpha \phi^\alpha_A = \frac{\partial \phi^i}{\partial t^A}, \qquad 0 = \frac{\partial \phi^\alpha_A}{\partial t^B} - \frac{\partial \phi^\alpha_B}{\partial t^A} + \mathcal{C}^\alpha_{\beta\gamma} \phi^\beta_B \phi^\alpha_A. \tag{4.25}$$

Remark 4.16. In the standard case ($E = TQ$), the morphism conditions reduce to

$$\phi^i_A = \frac{\partial \phi^i}{\partial t^A} \qquad \text{and} \qquad \frac{\partial \phi^i_A}{\partial t^B} = \frac{\partial \phi^i_B}{\partial t^A},$$

i.e. in considering morphisms we are considering the first-order prolongation of fields $\phi : \mathbb{R}^k \to Q$.

The Euler–Lagrange equations. Given a regular Lagrangian function $L : \overset{k}{\oplus} E \to \mathbb{R}$, it is natural to consider sections $\xi = (\xi_1, \ldots, \xi_k)$ of $(\mathcal{T}^E)^1_k(\overset{k}{\oplus} E) = \mathcal{T}^E(\overset{k}{\oplus} E) \oplus \overset{k}{\cdots} \oplus \mathcal{T}^E(\overset{k}{\oplus} E) \to \overset{k}{\oplus} E$ such that

$$\sum_{A=1}^k i_{\xi_A} \Omega^A_L = \mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E)} E_L, \tag{4.26}$$

equation (4.26) being the analog of the geometric Euler–Lagrange equations of the standard k-symplectic Lagrangian formalism.

Each ξ_A here is a section of the Lagrangian prolongation $\mathcal{T}^E(\overset{k}{\oplus} E)$, and with respect to a local coordinate system $\left(q^i, y^\alpha_A\right)$ on $\overset{k}{\oplus} E$ and a local basis $\{e_\alpha\}$ of $\mathrm{Sec}(E)$ it is given locally by

$$\xi_A = \xi^\alpha_A \mathcal{X}_\alpha + (\xi_A)^\alpha_C \mathcal{V}^C_\alpha.$$

Hence, by (4.6), (4.23) and (4.24), equation (4.26) is expressed locally as follows:

$$\xi^\beta_A \left(\rho^i_\alpha \frac{\partial^2 L}{\partial q^i \partial y^\beta_A} - \rho^i_\beta \frac{\partial^2 L}{\partial q^i \partial y^\alpha_A} + \mathcal{C}^\gamma_{\beta\alpha} \frac{\partial L}{\partial y^\gamma_A} \right) - (\xi_A)^\beta_B \frac{\partial^2 L}{\partial y^\beta_B \partial y^\alpha_A} = \rho^i_\alpha \left(y^\beta_A \frac{\partial^2 L}{\partial q^i \partial y^\beta_A} - \frac{\partial L}{\partial q^i} \right),$$

$$\xi^\alpha_A \frac{\partial^2 L}{\partial y^\beta_B \partial y^\alpha_A} = y^\alpha_A \frac{\partial^2 L}{\partial y^\beta_B \partial y^\alpha_A}.$$

Since L is regular, that is the matrix $\left(\frac{\partial^2 L}{\partial y^\alpha_A \partial y^\beta_B}\right)$ is regular, the above equations reduce to

$$\begin{aligned} & y^\beta_A \rho^i_\beta \frac{\partial^2 L}{\partial q^i \partial y^\alpha_A} + (\xi_A)^\beta_B \frac{\partial^2 L}{\partial y^\alpha_A \partial y^\beta_B} = \rho^i_\alpha \frac{\partial L}{\partial q^i} + y^\beta_A \mathcal{C}^\gamma_{\beta\alpha} \frac{\partial L}{\partial y^\gamma_A}, \\ & \xi^\alpha_A = y^\alpha_A. \end{aligned} \tag{4.27}$$

Thus ξ is a SOPDE. If $\widetilde{\Phi} : \mathbb{R}^k \to \overset{k}{\oplus} E$, the map associated with a Lie algebroid morphism, $\Phi : T\mathbb{R}^k \to E$, is such that $\widetilde{\Phi}(\mathbf{t}) = (\phi^i(\mathbf{t}), \phi^\alpha_A(\mathbf{t}))$ is an integral section of ξ, then by condition (4.19) and equations (4.27) we obtain

$$\begin{aligned}
&\frac{\partial \phi^i}{\partial t^A}\bigg|_{\mathbf{t}} \frac{\partial^2 L}{\partial q^i \partial y^\alpha_A}\bigg|_{\widetilde{\Phi}(\mathbf{t})} + \frac{\partial \phi^\beta_B}{\partial t^A}\bigg|_{\mathbf{t}} \frac{\partial^2 L}{\partial y^\alpha_A \partial y^\beta_B}\bigg|_{\widetilde{\Phi}(\mathbf{t})} = \rho^i_\alpha \frac{\partial L}{\partial q^i}\bigg|_{\widetilde{\Phi}(\mathbf{t})} + \phi^\beta_A C^\gamma_{\beta\alpha} \frac{\partial L}{\partial y^\gamma_A}\bigg|_{\widetilde{\Phi}(\mathbf{t})},\\
&\frac{\partial \phi^i}{\partial t^A}\bigg|_{\mathbf{t}} = \rho^i_\alpha \phi^\alpha_A(\mathbf{t}),\\
&0 = \frac{\partial \phi^\alpha_A}{\partial t^B}\bigg|_{\mathbf{t}} - \frac{\partial \phi^\alpha_B}{\partial t^A}\bigg|_{\mathbf{t}} + C^\alpha_{\beta\gamma} \phi^\beta_B(\mathbf{t}) \phi^\gamma_A(\mathbf{t}),
\end{aligned}$$

where the last equation is a consequence of the morphism condition (4.25). These equations can also be written in the form

$$\begin{aligned}
&\sum_{A=1}^k \frac{\partial}{\partial t^A}\left(\frac{\partial L}{\partial y^\alpha_A}\bigg|_{\widetilde{\Phi}(\mathbf{t})}\right) = \rho^i_\alpha \frac{\partial L}{\partial q^i}\bigg|_{\widetilde{\Phi}(\mathbf{t})} + \phi^\beta_C C^\gamma_{\beta\alpha} \frac{\partial L}{\partial y^\gamma_C}\bigg|_{\widetilde{\Phi}(\mathbf{t})}\\
&\frac{\partial \phi^i}{\partial t^A}\bigg|_{\mathbf{t}} = \rho^i_\alpha \phi^\alpha_A(\mathbf{t}),\\
&0 = \frac{\partial \phi^\alpha_A}{\partial t^B}\bigg|_{\mathbf{t}} - \frac{\partial \phi^\alpha_B}{\partial t^A}\bigg|_{\mathbf{t}} + C^\alpha_{\beta\gamma} \phi^\beta_B(\mathbf{t}) \phi^\gamma_A(\mathbf{t}).
\end{aligned} \tag{4.28}$$

If E is the standard Lie algebroid TQ, then these are the classical Euler–Lagrange equations of field theory for the Lagrangian $L : T^1_k Q \to \mathbb{R}$. In what follows (4.28) will be called the *Euler-Lagrange equations of field theories on Lie algebroids*.

Remark 4.17.

(i) Equations (4.28) are obtained by Martínez [33] using a variational approach in the multisymplectic framework.
(ii) When $k = 1$, equations (4.28) are the Euler–Lagrange equations on Lie algebroids given by Weinstein [47].
(iii) When $E = TQ$, equations (4.28) coincide with the Euler–Lagrange equations of the Günther formalism [36].

The results of this section can be summarized in the following.

Theorem 4.18. *Let $L : \overset{k}{\oplus} E \to \mathbb{R}$ be a regular Lagrangian, and $\xi_1, \ldots, \xi_k$ be k sections of $\widetilde{\tau}_{\overset{k}{\oplus} E} : \mathcal{T}^E(\overset{k}{\oplus} E) \to \overset{k}{\oplus} E$, such that*

$$\sum_{A=1}^k i_{\xi_A} \Omega^A_L = d^{\mathcal{T}^E(\overset{k}{\oplus} E)} E_L.$$

Then

(i) $\xi = (\xi_1, \ldots, \xi_k)$ is a SOPDE.
(ii) If $\widetilde{\Phi} : \mathbb{R}^k \to \overset{k}{\oplus} E$ is the map associated with a Lie algebroid morphism between $T\mathbb{R}^k$ and E, and is an integral section of ξ, then it is a solution of the Euler–Lagrange equations of field theories on Lie algebroids (4.28).

Remark 4.19. Rewriting this section for $k = 1$ affords Lagrangian mechanics on Lie algebroid (see section 3.1 of [8] or section 2.2 of [19]).

Finally we point out that the standard Lagrangian k-symplectic formalism is that particular case of the Lagrangian formalism on Lie algebroids in which $E = TQ$, the anchor map ρ_{TQ} is the identity on TQ and the structure constants $\mathcal{C}_{\alpha\beta}^{\gamma} = 0$. In this case

- the manifold $\overset{k}{\oplus} E$ is $T_k^1 Q$, $\mathcal{T}^{TQ}(T_k^1 Q)$ is $T(T_k^1 Q)$ and $(\mathcal{T}^{TQ})_k^1(T_k^1 Q)$ is $T_k^1(T_k^1 Q)$;
- the energy function $E_L : T_k^1 Q \to \mathbb{R}$ is given by $E_L = \sum_{A=1}^k \Delta_A(L) - L$, where Δ_A are the canonical vector fields on $T_k^1 Q$ (see remark 4.5);
- a section $\xi : \overset{k}{\oplus} E \to (\mathcal{T}^E)_k^1(\overset{k}{\oplus} E)$ is a k-vector field $\xi = (\xi_1, \ldots, \xi_k)$ on $T_k^1 Q$, that is ξ is a section of $\tau_{T_k^1 Q}^k : T_k^1(T_k^1 Q) \to T_k^1 Q$;
- a SOPDE ξ is a k-vector field on $T_k^1 Q$ that is a section of $T_k^1(\tau_Q^k) : T_k^1(T_k^1 Q) \to T_k^1 Q$;
- if f is a function on $T_k^1 Q$, then
$$\mathrm{d}^{\mathcal{T}^{TQ}(T_k^1 Q)} f(Y) = \mathrm{d}f(Y),$$
where $\mathrm{d}f$ denotes the standard exterior derivative and Y is a vector field on $T_k^1 Q$;
- $\Omega_L^A(X, Y) = \omega_L^A(X, Y)$, where ω_L^A, $A = 1, \ldots, k$, are the Lagrangian 2-forms of the standard k-symplectic formalism, given by $\omega_L^A = -\mathrm{d}(\mathrm{d}L \circ J^A)$;
- equation (4.26) can be written in the form
$$\sum_{A=1}^k i_{\xi_A} \omega_L^A = \mathrm{d}E_L,$$
that is as the geometric Euler–Lagrange equations of the standard Lagrangian k-symplectic formalism;
- the map $\widetilde{\Phi}$ associated with the Lie algebroid morphism $(T\phi, \phi)$ between $T\mathbb{R}^k$ and TQ that is induced by a map $\phi : \mathbb{R}^k \to Q$ is the first prolongation $\phi^{(1)}$ of ϕ :
$$\widetilde{\Phi}(\mathbf{t}) = \left(T\phi\left(\frac{\partial}{\partial t^1}\bigg|_{\mathbf{t}} \right), \ldots, T\phi\left(\frac{\partial}{\partial t^k}\bigg|_{\mathbf{t}} \right) \right)$$
(see definition 2.2).

Thus, by theorem 4.18, the following corollary summarizes the standard Lagrangian k-symplectic formalism [14, 36, 41].

Corollary 4.20. *Let $L : T_k^1 Q \to \mathbb{R}$ be a regular Lagrangian and $\xi = (\xi_1, \ldots, \xi_k)$ a k-vector field on $T_k^1 Q$ such that*
$$\sum_{A=1}^k i_{\xi_A} \omega_L^A = \mathrm{d}E_L.$$
Then

(i) ξ is a SOPDE*;*
(ii) if $\widetilde{\Phi} \equiv \phi^{(1)}$ is an integral section of the k-vector field ξ, then it is a solution of the Euler–Lagrange field equations of the standard Lagrangian k-symplectic field theory,
$$\sum_{A=1}^k \frac{\partial}{\partial t^A}\bigg|_{\mathbf{t}} \left(\frac{\partial L}{\partial v_A^i}\bigg|_{\widetilde{\Phi}(\mathbf{t})} \right) = \frac{\partial L}{\partial q^i}\bigg|_{\widetilde{\Phi}(\mathbf{t})}, \qquad v_A^i(\widetilde{\Phi}(\mathbf{t})) = \frac{\partial (q^i \circ \widetilde{\Phi})}{\partial t^A}\bigg|_{\mathbf{t}}.$$

4.2. Hamiltonian formalism

In this subsection we extend the standard Hamiltonian k-symplectic formalism to Lie algebroids. Throughout, we consider a Lie algebroid $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ over a manifold Q, and the dual bundle, $\tau^* : E^* \to Q$ of E.

4.2.1. The manifold $\overset{k}{\oplus} E^*$. The arena of the standard Hamiltonian k-symplectic formalism is the bundle $(T^1_k)^* Q$ of k^1-covelocities of Q, that is the Whitney sum of k copies of T^*Q. In generalizing the theory to Lie algebroids it is natural to consider that the analog of $(T^1_k)^* Q$ is $\overset{k}{\oplus} E^* = E^* \oplus \overset{k}{\cdots} \oplus E^*$, the Whitney of k copies of the dual space E^* with the projection map

$$\widetilde{\tau}^* : \overset{k}{\oplus} E^* \to Q, \qquad \widetilde{\tau}^*(a^*_{1_q}, \ldots, a^*_{k_q}) = q.$$

If (q^i, y_α) are local coordinates on $(\tau^*)^{-1}(U) \subseteq E^*$, then the induced local coordinates (q^i, y^A_α) on $(\widetilde{\tau}^*)^{-1}(U) \subseteq \overset{k}{\oplus} E^*$ are given by

$$q^i\big(a^*_{1_q}, \ldots, a^*_{k_q}\big) = q^i(q), \qquad y^A_\alpha\big(a^*_{1_q}, \ldots, a^*_{k_q}\big) = y_\alpha\big(a^*_{A_q}\big).$$

4.2.2. The Hamiltonian prolongation. We next consider the prolongation of a Lie algebroid E over the fibration $\widetilde{\tau}^*$: $\overset{k}{\oplus} E^* \to Q$, that is (see section 3.4)

$$\mathcal{T}^E(\overset{k}{\oplus} E^*) = \big\{\big(e_q, v_{\mathbf{b}^*_q}\big) \in E \times T(\overset{k}{\oplus} E^*) / \rho_E(e_q) = T(\widetilde{\tau}^*)\big(v_{\mathbf{b}^*_q}\big)\big\}. \tag{4.29}$$

Taking into account the description of the prolongation $\mathcal{T}^E P$ and the results of section 3.4 (see also [8, 19, 30]), we obtain

(i) $\mathcal{T}^E(\overset{k}{\oplus} E^*) \equiv E \times_{TQ} T(\overset{k}{\oplus} E^*)$ is a Lie algebroid over $\overset{k}{\oplus} E^*$ with the projection

$$\widetilde{\tau}_{\overset{k}{\oplus} E^*} : \mathcal{T}^E(\overset{k}{\oplus} E^*) \equiv E \times_{TQ} T(\overset{k}{\oplus} E^*) \longrightarrow \overset{k}{\oplus} E^*,$$

and Lie algebroid structure $([\![\cdot,\cdot]\!]^{\widetilde{\tau}^*}, \rho^{\widetilde{\tau}^*})$, where the anchor map

$$\rho^{\widetilde{\tau}^*} = E \times_{TQ} T(\overset{k}{\oplus} E^*) : \mathcal{T}^E(\overset{k}{\oplus} E^*) \to T(\overset{k}{\oplus} E^*)$$

is the canonical projection onto the second factor. We refer to this Lie algebroid as the *Hamiltonian prolongation*.

(ii) Local coordinates $\big(q^i, y^A_\alpha\big)$ on $\overset{k}{\oplus} E^*$ induce local coordinates $\big(q^i, y^A_\alpha, z^\alpha, w^A_\alpha\big)$ on $\mathcal{T}^E(\overset{k}{\oplus} E^*) \equiv E \times_{TQ} T(\overset{k}{\oplus} E^*)$, where

$$\begin{aligned} q^i\big(e_q, v_{\mathbf{b}^*_q}\big) &= q^i(q), & y^A_\alpha\big(e_q, v_{\mathbf{b}^*_q}\big) &= y^A_\alpha\big(\mathbf{b}^*_q\big), \\ z^\alpha\big(e_q, v_{\mathbf{b}^*_q}\big) &= y^\alpha(e_q), & w^A_\alpha\big(e_q, v_{\mathbf{b}^*_q}\big) &= v_{\mathbf{b}^*_q}\big(y^A_\alpha\big). \end{aligned} \tag{4.30}$$

(iii) The set $\{\mathcal{X}_\alpha, \mathcal{V}^\alpha_A\}$ given by

$$\begin{array}{rccl} \mathcal{X}_\alpha : & \overset{k}{\oplus} E^* & \to & \mathcal{T}^E(\overset{k}{\oplus} E^*) \equiv E \times_{TQ} T(\overset{k}{\oplus} E^*) \\ & \mathbf{b}^*_q & \mapsto & \mathcal{X}_\alpha(\mathbf{b}^*_q) = \left(e_\alpha(q); \rho^i_\alpha(q) \left.\dfrac{\partial}{\partial q^i}\right|_{\mathbf{b}^*_q}\right) \\ \mathcal{V}^\alpha_A : & \overset{k}{\oplus} E^* & \to & \mathcal{T}^E(\overset{k}{\oplus} E^*) \equiv E \times_{TQ} T(\overset{k}{\oplus} E^*) \\ & \mathbf{b}^*_q & \mapsto & \mathcal{V}^\alpha_A(\mathbf{b}^*_q) = \left(0_q; \left.\dfrac{\partial}{\partial y^A_\alpha}\right|_{\mathbf{b}^*_q}\right), \end{array} \tag{4.31}$$

is a local basis of $\mathrm{Sec}(\mathcal{T}^E(\overset{k}{\oplus} E^*))$, the set of sections of $\widetilde{\tau}_{\overset{k}{\oplus} E^*}$ (see (3.7)).

(iv) The anchor map $\rho^{\widetilde{\tau}^*} : \mathcal{T}^E(\overset{k}{\oplus} E^*) \to T(\overset{k}{\oplus} E^*)$ allows us to associate a vector field with each section $\xi \colon \overset{k}{\oplus} E^* \to \mathcal{T}^E(\overset{k}{\oplus} E^*)$ of $\widetilde{\tau}_{\overset{k}{\oplus} E^*}$. Locally, if ξ is given by

$$\xi = \xi^\alpha \mathcal{X}_\alpha + \xi^A_\alpha \mathcal{V}^\alpha_A \in \mathrm{Sec}(\mathcal{T}^E(\overset{k}{\oplus} E^*)),$$

then the associated vector field is

$$\rho^{\widetilde{\tau}^*}(\xi) = \rho^i_\alpha \xi^\alpha \frac{\partial}{\partial q^i} + \xi^A_\alpha \frac{\partial}{\partial y^A_\alpha} \in \mathfrak{X}(\overset{k}{\oplus} E^*). \tag{4.32}$$

(v) The Lie bracket of two sections of $\widetilde{\tau}_{\overset{k}{\oplus} E^*}$ is characterized by the relations

$$[\![\mathcal{X}_\alpha, \mathcal{X}_\beta]\!]^{\widetilde{\tau}^*} = \mathcal{C}^\gamma_{\alpha\beta} \mathcal{X}_\gamma \quad [\![\mathcal{X}_\alpha, \mathcal{V}^\beta_B]\!]^{\widetilde{\tau}^*} = 0 \quad [\![\mathcal{V}^\alpha_A, \mathcal{V}^\beta_B]\!]^{\widetilde{\tau}^*} = 0 \tag{4.33}$$

(see (3.8)).

(vi) If $\{\mathcal{X}^\alpha, \mathcal{V}^A_\alpha\}$ is the dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}^\alpha_A\}$, then the exterior differential is given by

$$\begin{aligned} &\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E^*)} f = \rho^i_\alpha \frac{\partial f}{\partial q^i} \mathcal{X}^\alpha + \frac{\partial f}{\partial y^A_\alpha} \mathcal{V}^A_\alpha, &&\text{for all} \quad f \in \mathcal{C}^\infty(\overset{k}{\oplus} E^*) \\ &\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E^*)} \mathcal{X}^\gamma = -\frac{1}{2} \mathcal{C}^\gamma_{\alpha\beta} \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, &&\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E^*)} \mathcal{V}^A_\gamma = 0 \end{aligned} \tag{4.34}$$

(see (3.9)).

Remark 4.21. In the particular case $E = TQ$, the manifold $\mathcal{T}^E(\overset{k}{\oplus} E^*)$ reduces to $T\big((T^1_k)^* Q\big)$. The proof is analogous to that of remark 4.2.

4.2.3. The vector bundle $\mathcal{T}^E(\overset{k}{\oplus} E^*) \oplus \overset{k}{\cdots} \oplus \mathcal{T}^E(\overset{k}{\oplus} E^*)$. In the standard Hamiltonian k-symplectic formalism one obtains the solutions of the Hamilton equations as integral sections of certain k-vector fields on $(T^1_k)^* Q$, that is certain sections of

$$\tau^k_{(T^1_k)^* Q} : T^1_k\big((T^1_k)^* Q\big) \to (T^1_k)^* Q.$$

Since on Lie algebroids $\mathcal{T}^E(\overset{k}{\oplus} E^*)$ plays the role of $T\big((T^1_k)^* Q\big)$, it is natural to assume that the role of

$$T^1_k\big((T^1_k)^* Q\big) = T\big((T^1_k)^* Q\big) \oplus \overset{k}{\cdots} \oplus T\big((T^1_k)^* Q\big)$$

is played by

$$(\mathcal{T}^E)^1_k(\overset{k}{\oplus} E^*) := \mathcal{T}^E(\overset{k}{\oplus} E^*) \oplus \overset{k}{\cdots} \oplus \mathcal{T}^E(\overset{k}{\oplus} E^*),$$

the Whitney sum of k copies of $\mathcal{T}^E(\overset{k}{\oplus} E^*)$ with canonical projection $\widetilde{\tau}^k_{\overset{k}{\oplus} E^*} : (\mathcal{T}^E)^1_k(\overset{k}{\oplus} E^*) \to \overset{k}{\oplus} E^*$ given by

$$\widetilde{\tau}^k_{\overset{k}{\oplus} E^*}\big(Z^1_{\mathbf{b}^*_q}, \ldots, Z^k_{\mathbf{b}^*_q}\big) = \mathbf{b}^*_q,$$

where $Z^A_{\mathbf{b}^*_q} = (a_{Aq}, v_{A\mathbf{b}^*_q}) \in \mathcal{T}^E(\overset{k}{\oplus} E^*)$, $A = 1, \ldots, k$. We now prove that there exists a k-vector field on $\overset{k}{\oplus} E^*$ associated with each section ξ of $\widetilde{\tau}^k_{\overset{k}{\oplus} E^*}$. Note that to give a section

$$\xi \colon \overset{k}{\oplus} E^* \to (\mathcal{T}^E)^1_k(\overset{k}{\oplus} E^*)) = \mathcal{T}^E(\overset{k}{\oplus} E^*) \oplus \overset{k}{\cdots} \oplus \mathcal{T}^E(\overset{k}{\oplus} E^*)$$

of $\widetilde{\tau}^k_{\overset{k}{\oplus}E^*}$ is equivalent to giving k sections $\xi_1, \ldots, \xi_k$ of the Hamiltonian prolongation $\mathcal{T}^E(\overset{k}{\oplus}E^*)$, namely the projections of ξ on each summand $\mathcal{T}^E(\overset{k}{\oplus}E^*)$.

Proposition 4.22. *Let* $\xi = (\xi^1, \ldots, \xi^k)$ *be a section of* $\widetilde{\tau}^k_{\overset{k}{\oplus}E^*}$. *Then*

$$(\rho^{\widetilde{\tau}^*}(\xi_1), \ldots, \rho^{\widetilde{\tau}^*}(\xi_k)) \colon \overset{k}{\oplus}E^* \to T^1_k(\overset{k}{\oplus}E^*)$$

is a k-vector field on $\overset{k}{\oplus}E^*$, *where* $\rho^{\widetilde{\tau}^*}$ *is the anchor map of the Lie algebroid* $\mathcal{T}^E(\overset{k}{\oplus}E^*)$.

Proof. Directly from (4.32) and the above remark. □

4.2.4. Hamiltonian formalism. Let $(E, [\![\cdot,\cdot]\!]_E, \rho_E)$ be a Lie algebroid on a manifold Q, and $H : \overset{k}{\oplus}E^* \to \mathbb{R}$ a Hamiltonian function. To develop the Hamiltonian k-symplectic formalism on Lie algebroids, we first generalize the Liouville forms of the standard case.

The Liouville sections *Liouville 1-sections* are defined to be sections of the bundle $(\mathcal{T}^E(\overset{k}{\oplus}E^*))^* \to \overset{k}{\oplus}E^*$ such that

$$\begin{array}{rccc} \Theta^A\colon & \overset{k}{\oplus} E^* & \longrightarrow & (\mathcal{T}^E(\overset{k}{\oplus} E^*))^* \\ & \mathbf{b}^*_q & \longmapsto & \Theta^A_{\mathbf{b}^*_q} \end{array} \qquad 1 \leqslant A \leqslant k,$$

where $\Theta^A_{\mathbf{b}^*_q} : (\mathcal{T}^E(\overset{k}{\oplus} E^*))_{\mathbf{b}^*_q} \longrightarrow \mathbb{R}$ is the function given by

$$\left(e_q, v_{\mathbf{b}^*_q}\right) \longmapsto \Theta^A_{\mathbf{b}^*_q}\left(e_q, v_{\mathbf{b}^*_q}\right) = b^*_{A_q}(e_q), \tag{4.35}$$

for each $e_q \in E$, $\mathbf{b}^*_q = \left(b_{1}{}^*_q, \ldots, b_{k}{}^*_q\right) \in \overset{k}{\oplus}E^*$ and $v_{\mathbf{b}^*_q} \in T_{\mathbf{b}^*_q}(\overset{k}{\oplus}E^*)$. Liouville 2-sections

$$\Omega^A : \overset{k}{\oplus} E^* \to (\mathcal{T}^E(\overset{k}{\oplus} E^*))^* \wedge (\mathcal{T}^E(\overset{k}{\oplus} E^*))^*, \qquad 1 \leqslant A \leqslant k$$

are defined by

$$\Omega^A = -\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus}E^*)}\Theta^A,$$

where $\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus}E^*)}$ denotes the exterior differential on the Lie algebroid $\mathcal{T}^E\left(\overset{k}{\oplus}E^*\right)$ (see (4.34)).

Locally, if $\left\{\mathcal{X}_\alpha, \mathcal{V}^\beta_B\right\}$ is a local basis of $\mathrm{Sec}(\mathcal{T}^E(\overset{k}{\oplus}E^*))$ and $\left\{\mathcal{X}^\alpha_A, \mathcal{V}^B_\beta\right\}$ its dual basis, then by (4.31),

$$\Theta^A = \sum_{\beta=1}^{m} y^A_\beta \mathcal{X}^\beta, \qquad 1 \leqslant A \leqslant k, \tag{4.36}$$

and by (4.32), (4.33), (4.34) and (4.36),

$$\Omega^A = \sum_\beta \mathcal{X}^\beta \wedge \mathcal{V}^A_\beta + \frac{1}{2}\sum_{\beta,\gamma,\delta} \mathcal{C}^\delta_{\beta\gamma} y^A_\delta \mathcal{X}^\beta \wedge \mathcal{X}^\gamma, \qquad 1 \leqslant A \leqslant k. \tag{4.37}$$

Remark 4.23.

(i) When $k = 1$, the Liouville sections introduced here are the Liouville sections of mechanics on Lie algebroids; see Martínez [8, 31].

(ii) When $E = TQ$ and $\rho_{TQ} = id_{TQ}$, then

$$\Omega^A(X,Y) = \omega^A(X,Y), \qquad 1 \leqslant A \leqslant k,$$

where X, Y are vector fields on $\left(T^1_k\right)^* Q$ and $\omega^1, \ldots, \omega^k$ are the canonical 2-forms of the standard Hamiltonian k-symplectic formalism.

The Hamilton equations.

Theorem 4.24. *Let* $H : \overset{k}{\oplus} E^* \to \mathbb{R}$ *be a Hamiltonian and*

$$\xi = (\xi_1, \ldots, \xi_k) : \overset{k}{\oplus} E^* \to (\mathcal{T}^E)^1_k(\overset{k}{\oplus} E^*) \equiv \mathcal{T}^E(\overset{k}{\oplus} E^*) \oplus \overset{k}{\cdots} \oplus \mathcal{T}^E(\overset{k}{\oplus} E^*)$$

a section of $\widetilde{\tau}^k_{\overset{k}{\oplus} E^*}$ *such that*

$$\sum_{A=1}^k i_{\xi_A} \Omega^A = d^{\mathcal{T}^E(\overset{k}{\oplus} E^*)} H. \tag{4.38}$$

If $\psi : \mathbb{R}^k \to \overset{k}{\oplus} E^*$ *is an integral section of* ξ*, then* ψ *is a solution of the following system of partial differential equations:*

$$\frac{\partial \psi^i}{\partial t^A} = \rho^i_\alpha \frac{\partial H}{\partial y^A_\alpha} \qquad \text{and} \qquad \sum_{A=1}^k \frac{\partial \psi^A_\alpha}{\partial t^A} = -\left(\mathcal{C}^\delta_{\alpha\beta}\, \psi^B_\delta \frac{\partial H}{\partial y^B_\beta} + \rho^i_\alpha \frac{\partial H}{\partial q^i} \right). \tag{4.39}$$

Remark 4.25. In the particular case $E = TQ$ and $\rho = \mathrm{id}_{TQ}$, equations (4.39) are the Hamilton field equations. Accordingly, equations (4.39) are called *the Hamilton equations for Lie algebroids*.

Proof. Consider $\left\{\mathcal{X}_\alpha, \mathcal{V}^\beta_B\right\}$, a local basis of sections of $\widetilde{\tau}_{\overset{k}{\oplus} E^*} : \mathcal{T}^E(\overset{k}{\oplus} E^*) \to \overset{k}{\oplus} E^*$. Each ξ_A in the statement of the theorem can be written in the form

$$\xi_A = \xi^\alpha_A \mathcal{X}_\alpha + (\xi_A)^B_\alpha \mathcal{V}^\alpha_B, \tag{4.40}$$

and by (4.34), (4.37) and (4.40) the local expression of (4.38) is

$$\begin{aligned} \xi^\alpha_B &= \frac{\partial H}{\partial y^B_\alpha} \\ \sum_{A=1}^k (\xi_A)^A_\alpha &= -\left(\mathcal{C}^\delta_{\alpha\beta}\, y^C_\delta \frac{\partial H}{\partial y^C_\beta} + \rho^i_\alpha \frac{\partial H}{\partial q^i} \right). \end{aligned} \tag{4.41}$$

Also, if $\psi : \mathbb{R}^k \to \overset{k}{\oplus} E^*$, $\psi(\mathbf{t}) = \left(\psi^i(\mathbf{t}), \psi^A_\alpha(\mathbf{t})\right)$ is an integral section of ξ, that is ψ is an integral section of $(\rho^{\widetilde{\tau}^*}(\xi_1), \ldots, \rho^{\widetilde{\tau}^*}(\xi_k))$, the associated k-vector field on $\overset{k}{\oplus} E^*$, then

$$\xi^\beta_A \rho^i_\beta = \frac{\partial \psi^i}{\partial t^A}, \qquad (\xi_A)^B_\beta = \frac{\partial \psi^B_\beta}{\partial t^A}. \tag{4.42}$$

By (4.41) and (4.42),

$$\frac{\partial \psi^i}{\partial t^A} = \frac{\partial H}{\partial y^A_\alpha} \rho^i_\alpha \qquad \text{and} \qquad \sum_{A=1}^k \frac{\partial \psi^A_\alpha}{\partial t^A} = -\left(\mathcal{C}^\delta_{\alpha\beta} \psi^A_\delta \frac{\partial H}{\partial y^A_\beta} + \rho^i_\alpha \frac{\partial H}{\partial q^i} \right).$$ □

Remark 4.26. When $k = 1$, this theorem summarizes the Hamiltonian mechanics on Lie algebroids (see section 3.2 of [8] or section 3.3 of [19]).

The standard Hamiltonian k-symplectic formalism is the particular case of the general formalism on Lie algebroids in which $E = TQ$ and $\rho_E = id_{TQ}$. Specifically in this case,

- the manifold $\overset{k}{\oplus} E^*$ is $\left(T^1_k\right)^* Q$, $\mathcal{T}^{TQ}\left(\left(T^1_k\right)^* Q\right)$ is $T\left(\left(T^1_k\right)^* Q\right)$ and $(\mathcal{T}^{TQ})^1_k\left(\left(T^1_k\right)^* Q\right)$ is $T^1_k\left(\left(T^1_k\right)^* Q\right)$;
- a section

$$\xi : \overset{k}{\oplus} E^* \to (\mathcal{T}^E)^1_k(\overset{k}{\oplus} E^*)$$

is a section of $\tau^k_{(T^1_k)^*Q} : T^1_k\left(\left(T^1_k\right)^* Q\right) \to \left(T^1_k\right)^* Q$, i.e. a k-vector field $\xi = (\xi_1, \dots, \xi_k)$ on $\left(T^1_k\right)^* Q$;
- if f is a function on $\left(T^1_k\right)^* Q$, then

$$(\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E^*)} f)(Y) = \mathrm{d}f(Y),$$

where $\mathrm{d}f$ denotes the usual exterior derivative and Y is a vector field on $\left(T^1_k\right)^* Q$;
- $$\Omega^A(X, Y) = \omega^A(X, Y) \quad (A = 1, \dots, k),$$

where the ω^A are the canonical k-symplectic 2-forms on $\left(T^1_k\right)^* Q$;
- equation (4.38) reduces to

$$\sum_{A=1}^{k} i_{\xi_A}\omega^A = dH$$

(so equation (4.38) is the geometric version of the Hamilton field equations of the standard k-symplectic formalism).

Accordingly, by theorem 4.24 the standard Hamiltonian k-symplectic formalism is summarized in the following.

Corollary 4.27. *Let $H : \left(T^1_k\right)^* Q \to \mathbb{R}$ be a Hamiltonian formalism and $\xi = (\xi_1, \dots, \xi_k)$ a k-vector field on $\left(T^1_k\right)^* Q$ such that*

$$\sum_{A=1}^{k} i_{\xi_A}\omega^A = dH.$$

If $\psi : \mathbb{R}^k \to \left(T^1_k\right)^* Q$, $\psi(\mathbf{t}) = \left(\psi^i(\mathbf{t}), \psi^A_i(\mathbf{t})\right)$ is an integral section of ξ, it is a solution of the Hamilton field equations in the standard k-symplectic formalism, that is

$$\sum_{A=1}^{k} \left.\frac{\partial \psi^A_i}{\partial t^A}\right|_{\mathbf{t}} = -\left.\frac{\partial H}{\partial q^i}\right|_{\psi(\mathbf{t})}, \qquad \left.\frac{\partial \psi^i}{\partial t^A}\right|_{\mathbf{t}} = \left.\frac{\partial H}{\partial p^A_i}\right|_{\psi(\mathbf{t})}, \qquad i = 1, \dots, n. \quad (4.43)$$

4.3. The Legendre transformation

In this section we define the Legendre transformation on Lie algebroids and establish the equivalence between the Lagrangian and Hamiltonian formalisms when the Lagrangian function is hyperregular.

Let $L : \overset{k}{\oplus} E \to \mathbb{R}$ be a Lagrangian function and $\Theta^A_L : \overset{k}{\oplus} E \to [\mathcal{T}^E(\overset{k}{\oplus} E)]^*$ $(A = 1, \dots, k)$ the Poincaré–Cartan 1-sections associated with L, as defined in (4.20).

Definition 4.28. *The Legendre transformation associated with L is the smooth map*

$$\mathfrak{Leg} : \overset{k}{\oplus} E \to \overset{k}{\oplus} E^*$$

defined by

$$\mathfrak{Leg}(b_{1_q},\ldots,b_{k_q}) = \left(\left[\mathfrak{Leg}(b_{1_q},\ldots,b_{k_q})\right]^1,\ldots,\left[\mathfrak{Leg}(b_{1_q},\ldots,b_{k_q})\right]^k\right),$$

where

$$\left[\mathfrak{Leg}(b_{1_q},\ldots,b_{k_q})\right]^A(e_q) = \frac{\mathrm{d}}{\mathrm{d}s}L(b_{1_q},\ldots,b_{Aq}+se_q,\ldots,b_{k_q})\Big|_{s=0}, \qquad e_q \in E_q.$$

In other words, for each A,

$$\left[\mathfrak{Leg}(b_{1_q},\ldots,b_{k_q})\right]^A(e_q) = \Theta_L^A(b_{1_q},\ldots,b_{k_q})(Z), \tag{4.44}$$

where Z is a point of the fiber of $(\mathcal{T}^E(\overset{k}{\oplus} E))_{\mathbf{b}_q}$ over the point

$$\mathbf{b}_q = (b_{1_q},\ldots,b_{k_q}) \in \overset{k}{\oplus} E,$$

such that

$$\widetilde{\tau}_1(Z) = e_q,$$

$\widetilde{\tau}_1 : \mathcal{T}^E(\overset{k}{\oplus} E) = E \times_{TQ} T(\overset{k}{\oplus} E) \to E$ being the projection on the first factor. Z is of the form $Z = (e_q, v_{b_q})$. The map $\mathfrak{Leg}$ is well defined, and its local expression is

$$\mathfrak{Leg}(q^i, y^\alpha_A) = \left(q^i, \frac{\partial L}{\partial y^\alpha_A}\right),$$

in view of which it is easy to prove that the Lagrangian L is regular if and only if $\mathfrak{Leg}$ is a local diffeomorphism.

Remark 4.29. When $E = TQ$, the Legendre transformation defined here coincides with the Legendre transformation introduced by Günther in [14].

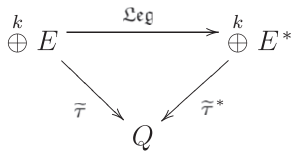
$\mathfrak{Leg}$ induces a map

$$\mathcal{T}^E\mathfrak{Leg} : \mathcal{T}^E(\overset{k}{\oplus} E) \equiv E \times_{TQ} T(\overset{k}{\oplus} E) \to \mathcal{T}^E(\overset{k}{\oplus} E^*) \equiv E \times_{TQ} T(\overset{k}{\oplus} E^*)$$

defined by

$$\mathcal{T}^E\mathfrak{Leg}(e_q, v_{\mathbf{b}_q}) = (e_q, (\mathfrak{Leg})_*(\mathbf{b}_q)(v_{\mathbf{b}_q})),$$

where $e_q \in E_q, \mathbf{b}_q \in \overset{k}{\oplus} E$ and $(e_q, v_{\mathbf{b}_q}) \in \mathcal{T}^E(\overset{k}{\oplus} E) \equiv E \times_{TQ} T(\overset{k}{\oplus} E)$. $\mathcal{T}^E\mathfrak{Leg}$ is well defined because the diagram



is commutative.

In local coordinates (see (4.2) and (4.30)),

$$\mathcal{T}^E\mathfrak{Leg}(q^i, y^\alpha_A, z^\alpha, w^\beta_B) = \left(q^i, \frac{\partial L}{\partial y^\alpha_A}, z^\alpha, z^\alpha\rho^i_\alpha\frac{\partial^2 L}{\partial q^i\partial y^\gamma_C} + w^\beta_B\frac{\partial^2 L}{\partial y^\gamma_C\partial y^\beta_B}\right). \tag{4.45}$$

Theorem 4.30. *The pair* $(\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})$ *is a morphism between the Lie algebroids* $(\mathcal{T}^E(\overset{k}{\oplus} E), \rho^{\widetilde{\tau}}, [\![\cdot,\cdot]\!]^{\widetilde{\tau}})$ *and* $(\mathcal{T}^E(\overset{k}{\oplus} E^*), \rho^{\widetilde{\tau}^*}, [\![\cdot,\cdot]\!]^{\widetilde{\tau}^*})$. *Moreover, if* Θ^A_L *and* Ω^A_L *are respectively the*

Poincaré–Cartan 1-sections and 2-sections associated with $L\colon \overset{k}{\oplus} E \to \mathbb{R}$, *and* Θ^A *and* (Ω^A) *the Liouville 1-sections and 2-sections on* $\mathcal{T}^E(\overset{k}{\oplus} E^*)$, *then*

$$(\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})^*\Theta^A = \Theta_L^A, \qquad (\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})^*\Omega^A = \Omega_L^A, \qquad 1 \leqslant A \leqslant k. \tag{4.46}$$

Proof. We first prove that $(\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})$ is a Lie algebroid morphism,

$$\begin{array}{ccc} \mathcal{T}^E(\overset{k}{\oplus} E) & \xrightarrow{\ \mathcal{T}^E\mathfrak{Leg}\ } & \mathcal{T}^E(\overset{k}{\oplus} E^*) \\ {\scriptstyle \tilde{\tau}_{\overset{k}{\oplus} E}}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\tau}_{\overset{k}{\oplus} E^*}} \\ \overset{k}{\oplus} E & \xrightarrow{\ \mathfrak{Leg}\ } & \overset{k}{\oplus} E^* \end{array}$$

Let (q^i) be local coordinates on Q, $\{e_\alpha\}$ a local basis of $\mathrm{Sec}(E)$, and $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha^A\}$ and $\{\mathcal{Y}_\alpha, \mathcal{U}_A^\alpha\}$ the corresponding local bases of sections of $\tilde{\tau}_{\overset{k}{\oplus} E} : \mathcal{T}^E(\overset{k}{\oplus} E) \to \overset{k}{\oplus} E$ and $\tilde{\tau}_{\overset{k}{\oplus} E^*} : \mathcal{T}^E(\overset{k}{\oplus} E^*) \to \overset{k}{\oplus} E^*$). Then by (3.5), (4.6) and (4.45), straightforward computation shows that

$$(\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})^*(\mathcal{Y}^\alpha) = \mathcal{X}^\alpha \qquad \text{and} \qquad (\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})^*\big(\mathcal{U}_\alpha^A\big) = \mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E)}\left(\frac{\partial L}{\partial y_A^\alpha}\right) \tag{4.47}$$

for each $\alpha = 1, \ldots, m$ and $A = 1, \ldots, k$, where $\{\mathcal{X}^\alpha, \mathcal{V}_A^\alpha\}$ and $\{\mathcal{Y}^\alpha, \mathcal{U}_\alpha^A\}$ are the dual bases of $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha^A\}$ and $\{\mathcal{Y}_\alpha, \mathcal{U}_A^\alpha\}$, respectively. By (4.6) and (4.34) we therefore conclude that

$$\begin{aligned} (\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})^*\big(\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E^*)} f\big) &= \mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E)}(f \circ \mathfrak{Leg}) \\ (\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})^*\big(\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E^*)} \mathcal{Y}^\alpha\big) &= \mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E)}((\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})^*\mathcal{Y}^\alpha) \\ (\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})^*\big(\mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E^*)} \mathcal{U}_\alpha^A\big) &= \mathrm{d}^{\mathcal{T}^E(\overset{k}{\oplus} E)}\big((\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})^*\mathcal{U}_\alpha^A\big) \end{aligned}$$

for all functions $f \in \mathcal{C}^\infty(\overset{k}{\oplus} E^*)$ and all α and A. Consequently, $(\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})$ is a Lie algebroid morphism.

To show that $(\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})^*\Theta^A = \Theta_L^A$, we note that by (3.5), (4.35) and (4.44),

$$\begin{aligned} [(\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})^*\Theta^A]_{\mathbf{b}_q}\big(e_q, v_{\mathbf{b}_q}\big) &= \Theta^A_{\mathfrak{Leg}(\mathbf{b}_q)}\big(e_q, (\mathfrak{Leg})_*(\mathbf{b}_q)\big(v_{\mathbf{b}_q}\big)\big) \\ &= [\mathfrak{Leg}(\mathbf{b}_q)]^A(e_q) = \Theta_L^A(\mathbf{b}_q)\big(e_q, v_{\mathbf{b}_q}\big). \end{aligned}$$

Finally, since $(\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})$ is a Lie algebroid morphism, this result for 1-sections implies that

$$(\mathcal{T}^E\mathfrak{Leg}, \mathfrak{Leg})^*\Omega^A = \Omega_L^A.$$

□

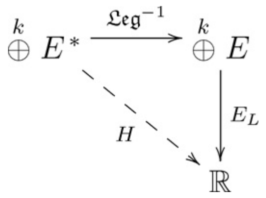
Remark 4.31.

(i) When $k = 1$, this theorem reduces to theorem 3.12 of [19].
(ii) When $E = TQ$ and $\rho_{TQ} = id_{TQ}$, it establishes the relationship between the Lagrangian and Hamiltonian formalisms in the standard *k*-symplectic approach.

We next assume that L is *hyperregular*, that is, that $\mathfrak{Leg}$ is a global diffeomorphism. In this case we may consider the Hamiltonian function $H : \overset{k}{\oplus} E^* \to \mathbb{R}$ defined by

$$H = E_L \circ (\mathfrak{Leg})^{-1},$$

where E_L is the Lagrangian energy associated with L, given by (4.24), and $(\mathfrak{Leg})^{-1}$ is the inverse of the Legendre transformation:



Lemma 4.32. *If the Lagrangian L is hyperregular, then $\mathcal{T}^E\mathfrak{Leg}$ is a diffeomorphism.*

Proof. Since in this case $\mathfrak{Leg}$ is a global diffeomorphism so that there exists an inverse map $\mathfrak{Leg}^{-1}\colon \overset{k}{\oplus} E^* \to \overset{k}{\oplus} E$, $\mathcal{T}^E\mathfrak{Leg}$ has the differentiable inverse

$$(\mathcal{T}^E\mathfrak{Leg})^{-1} : \mathcal{T}^E(\overset{k}{\oplus} E^*) \to \mathcal{T}^E(\overset{k}{\oplus} E)$$

given by

$$(\mathcal{T}^E\mathfrak{Leg})^{-1}\big(a_q, v_{\mathbf{b}_q^*}\big) = \big(a_q, (\mathfrak{Leg}^{-1})_*\big(\mathbf{b}_q^*\big)\big(v_{\mathbf{b}_q^*}\big)\big),$$

where $a_q \in E, \mathbf{b}_q^* \in \overset{k}{\oplus} E^*$ and $\big(a_q, v_{\mathbf{b}_q^*}\big) \in \mathcal{T}^E(\overset{k}{\oplus} E^*) \equiv E \times_{TQ} T(\overset{k}{\oplus} E^*)$. □

The following theorem establishes the equivalence between the Lagrangian and Hamiltonian k-symplectic formulations on Lie algebroids.

Theorem 4.33. *Let L be a hyperregular Lagrangian. There is a bijective correspondence between the set $\{\eta : \mathbb{R}^k \to \overset{k}{\oplus} E \mid \eta$ is an integral section of a solution ξ_L of the geometric Euler–Lagrange equations (4.26)$\}$ and the set $\{\psi : \mathbb{R}^k \to \overset{k}{\oplus} E^* \mid \psi$ is an integral section of some solution ξ_H of the geometric Hamilton equations (4.38)$\}$.*

Proof. The proof is similar to the standard case: see [46]. Essentially, if $\xi_L = \big(\xi_L^1, \ldots, \xi_L^k\big)\colon \overset{k}{\oplus} E \to (\mathcal{T}^E)^1_k(\overset{k}{\oplus} E)$ is a solution of the geometric Euler–Lagrange equations for Lie algebroids (4.26), then $\xi_H = \big(\xi_H^1, \ldots, \xi_H^k\big)$ is a solution of (4.38), where

$$\xi_H^A = \mathcal{T}^E\mathfrak{Leg} \circ \xi_L^A \circ (\mathfrak{Leg})^{-1}.$$

Moreover, *if $\eta : \mathbb{R}^k \to \overset{k}{\oplus} E$ is an integral section of $\xi_L = \big(\xi_L^1, \ldots, \xi_L^k\big)$, then*

$$\mathfrak{Leg} \circ \eta : \mathbb{R}^k \to \overset{k}{\oplus} E^*$$

is an integral section of $\xi_H = \big(\xi_H^1, \ldots, \xi_H^k\big)$.

The converse is proved similarly. □

Remark 4.34. The case $k = 1$ shows the equivalence between the Lagrangian and Hamiltonian forms of autonomous mechanics on Lie algebroids (see, for instance, [8]) and the case $E = TQ, \rho_{TQ} = id_{TQ}$, the equivalence between the Lagrangian and Hamiltonian formulations in the standard k-symplectic framework (see [46]).

5. Examples

Harmonic mappings [5, 6, 44, 48]. In this example we consider the harmonic mappings from $\mathbb{R}^2$ into a Lie group G. The harmonic mapping Lagrangian is given [44] by

$$L(\phi, \phi_x, \phi_y) = \tfrac{1}{2}\langle \phi^{-1}\phi_x, \phi^{-1}\phi_x\rangle + \tfrac{1}{2}\langle \phi^{-1}\phi_y, \phi^{-1}\phi_y\rangle, \tag{5.1}$$

where $\langle\cdot,\cdot\rangle$ is the *Killing form* on $\mathfrak{g}$ and ϕ_x, ϕ_y are the partial derivatives of $\phi : \mathbb{R}^2 \to G$ with respect to the local coordinates (x, y) of $\mathbb{R}^2$. The associated field equation is $\tau(\phi) = 0$, where $\tau(\phi)$ is the tension of ϕ, defined for general smooth mappings by

$$\tau(\phi)^i = h^{AB}\left(\frac{\partial^2\phi^i}{\partial t^A \partial t^B} - \Gamma^C_{AB}\frac{\partial\phi^i}{\partial t^C} + \mathcal{C}^i_{jk}\frac{\partial\phi^j}{\partial t^A}\frac{\partial\phi^k}{\partial t^B}\right), \qquad i = 1, \ldots, \dim G,$$

the h_{AB} being the components of a metric on $\mathbb{R}^2$ with Christoffel symbols Γ^C_{AB}, and $\mathcal{C}^i_{jk}$ the Christoffel symbols of the bi-invariant metric on G. In our case, h_{AB} is of course just the flat Euclidian metric.

Here we deal with the case $G = SO(3)$, considered as embedded in $\mathfrak{gl}(3)$, in which case the Killing form $\langle\cdot,\cdot\rangle$ is just the trace,

$$\langle\xi, \eta\rangle = -\mathrm{trace}(\xi\eta),$$

and the Lagrangian (5.1) is a function $L : TSO(3) \oplus TSO(3) = T^1_2(SO(3)) \to \mathbb{R}$.

Since $T^1_2(SO(3)) \cong SO(3) \times \mathfrak{so}(3) \times \mathfrak{so}(3)$, we identify $T^1_2(SO(3))/SO(3)$ with $\mathfrak{so}(3) \times \mathfrak{so}(3)$, and we consider the projection l of L to $\mathfrak{so}(3) \times \mathfrak{so}(3)$ given by

$$l(\xi_1, \xi_2) = -\tfrac{1}{2}\mathrm{trace}\left(\xi_1^2\right) - \tfrac{1}{2}\mathrm{trace}\left(\xi_2^2\right), \qquad \xi_1, \xi_2 \in \mathfrak{so}(3).$$

If $\{E_1, E_2, E_3\}$ is a basis of $\mathfrak{so}(3)$, so that $\xi_i = y_i^\alpha E_\alpha$ $(i = 1, 2)$, l is given locally by

$$l\left(y_1^\alpha, y_2^\alpha\right) = \sum_{\alpha=1}^{3}\left(\left(y_1^\alpha\right)^2 + \left(y_2^\alpha\right)^2\right).$$

Since a Lie algebra is an example of a Lie algebroid, we can apply the theory developed in section 4.1. The Euler–Lagrange equations (4.28) in this case are

$$\left.\begin{aligned}&\frac{\partial y_1^\alpha}{\partial t^1} + \frac{\partial y_2^\alpha}{\partial t^2} = 0\\ &\frac{\partial y_A^\alpha}{\partial t^B} - \frac{\partial y_B^\alpha}{\partial t^A} + \mathcal{C}^\alpha_{\beta\gamma}y_B^\beta y_A^\gamma = 0.\end{aligned}\right\} \qquad (\alpha = 1, 2, 3;\ A = 1, 2)$$

The Poisson-sigma model. Consider a Poisson manifold (Q, Λ). Then the cotangent bundle T^*Q has a Lie algebroid structure with anchor map

$$\begin{aligned}\rho : T^*Q &\to TQ\\ \beta &\mapsto \Lambda(\beta, \cdot),\end{aligned}$$

and bracket

$$[\alpha, \beta] = i_{\rho(\alpha)}\,d\beta - i_{\rho(\beta)}\,d\alpha - d\Lambda(\alpha, \beta).$$

In local coordinates, the bivector Λ has the expression

$$\Lambda = \frac{1}{2}\Lambda^{ij}\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial q^j}.$$

We can consider the Lagrangian for the Poisson-sigma model as a function on $T^*Q \oplus T^*Q$. Thus, if $\left(q^i, p_1^i, p_2^i\right)$ denotes local coordinates on $T^*Q \oplus T^*Q$, the local expression of the Lagrangian is (see [32])

$$L = -\tfrac{1}{2}\Lambda^{ij}p_i^1 p_j^2.$$

A long but straightforward calculation shows that the Euler–Lagrange equations (4.28) in this case are

$$\frac{1}{2}\Lambda^{ij}\left(\frac{\partial p_i^2}{\partial t^1} - \frac{\partial p_i^1}{\partial t^2} + \frac{\partial \Lambda^{kl}}{\partial q^i} p_k^1 p_l^2\right) = 0,$$
$$\frac{\partial q^i}{\partial t^A} + \Lambda^{ij} p_j^A = 0,$$
$$\frac{\partial p_i^2}{\partial t^1} - \frac{\partial p_i^1}{\partial t^2} + \frac{\partial \Lambda^{kl}}{\partial q^i} p_k^1 p_l^2 = 0.$$

However, in view of the morphism condition, the first equation vanishes. The solution of the remaining two is a field $\phi : \mathbb{R}^2 \to T^*Q \oplus T^*Q$ given locally by

$$\phi(\mathbf{t}) = \left(q^i(\mathbf{t}), p_i^1(\mathbf{t}), p_i^2(\mathbf{t})\right).$$

The conventional form of the field equations for the Poisson-sigma model [43] is

$$\mathrm{d}\phi^j + \Lambda^{jk} P_k = 0$$
$$\mathrm{d}P_j + \tfrac{1}{2}\Lambda^{kl}_{,j} P_k \wedge P_L = 0,$$

where the $P_j = p_j^1\,\mathrm{d}t^1 + p_j^2\,\mathrm{d}t^2$ $(j = 1, \ldots, n)$ are 1-forms on $\mathbb{R}^2$.

Remark 5.1. Poisson-sigma models were originally introduced by Schaller and Strobl [42] and Ikeda [16] so as to unify several two-dimensional models of gravity and cast them into a common form with Yang–Mills theories.

Systems with symmetry. Consider a principal bundle $\pi : \bar{Q} \longrightarrow Q = \bar{Q}/G$. Let $A : T\bar{Q} \longrightarrow \mathfrak{g}$ be a fixed principal connection with curvature $B : T\bar{Q} \oplus T\bar{Q} \longrightarrow \mathfrak{g}$. The connection A determines an isomorphism between the vector bundles $T\bar{Q}/G \to Q$ and $TQ \oplus \widetilde{\mathfrak{g}} \longrightarrow Q$, where $\widetilde{\mathfrak{g}} = (\bar{Q} \times \mathfrak{g})/G$ is the adjoint bundle (see [7]):

$$[v_{\bar{q}}] \leftrightarrow T_{\bar{q}}\pi(v_{\bar{q}}) \oplus [(\bar{q}, A(v_{\bar{q}}))],$$

where $v_{\bar{q}} \in T_{\bar{q}}\bar{Q}$. The connection allows us to obtain a local basis of sections of $\mathrm{Sec}(T\bar{Q}/G) = \mathfrak{X}(Q) \oplus \mathrm{Sec}(\widetilde{\mathfrak{g}})$ as follows. Let $\mathfrak{e}$ be the identity element of the Lie group G and assume that there are local coordinates (q^i), $1 \leqslant i \leqslant \dim Q$, and that $\{\xi_a\}$ is a basis of $\mathfrak{g}$. The corresponding sections of the adjoint bundle are the left-invariant vector fields ξ_a^L:

$$\xi_a^L(g) = T_{\mathfrak{e}} L_g(\xi_a),$$

where $L_g : G \longrightarrow G$ is left translation by $g \in G$. If

$$A\left(\left.\frac{\partial}{\partial q^i}\right|_{(q,\mathfrak{e})}\right) = A_i^a \xi_a,$$

then the corresponding horizontal lifts on the trivialization $U \times G$ are the vector fields

$$\left(\frac{\partial}{\partial q^i}\right)^h = \frac{\partial}{\partial q^i} - A_i^a \xi_a^L\,.$$

The elements of the set

$$\left\{\left(\frac{\partial}{\partial q^i}\right)^h, \xi_a^L\right\}$$

are by construction G-invariant, and therefore, constitute a local basis of sections $\{e_i, e_a\}$ of $\mathrm{Sec}(T\bar{Q}/G) = \mathfrak{X}(Q) \oplus \mathrm{Sec}(\widetilde{\mathfrak{g}})$.

Denote by (q^i, y^i, y^a) the induced local coordinates of $T\bar{Q}/G$. Then

$$B\left(\frac{\partial}{\partial q^i}_{(q,\mathfrak{e})}, \frac{\partial}{\partial q^j}_{(q,\mathfrak{e})}\right) = B^a_{ij}\xi_a,$$

where

$$B^c_{ij} = \frac{\partial A^c_i}{\partial q^j} - \frac{\partial A^c_j}{\partial q^i} - \mathcal{C}^c_{ab}A^a_i A^b_j,$$

the $\mathcal{C}^c_{ab}$ being the structure constants of the Lie algebra. The structure functions of the Lie algebroid $T\bar{Q}/G \to Q$ are determined (see [19]) by

$$\begin{aligned}
&[\![e_i, e_j]\!]_{T\bar{Q}/G} = -B^c_{ij}e_c\\
&[\![e_i, e_a]\!]_{T\bar{Q}/G} = \mathcal{C}^c_{ab}A^b_i e_c\\
&[\![e_a, e_b]\!]_{T\bar{Q}/G} = \mathcal{C}^c_{ab}e_c\\
&\rho_{T\bar{Q}/G}(e_i) = \frac{\partial}{\partial q^i}\\
&\rho_{T\bar{Q}/G}(e_a) = 0,
\end{aligned}$$

and for a Lagrangian function $L : \overset{k}{\oplus} T\bar{Q}/G \longrightarrow \mathbb{R}$ the Euler–Lagrange field equations are

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t^A}\left(\frac{\partial L}{\partial y^i_A}\right) = \frac{\partial L}{\partial q^i} + B^c_{ij}y^j_C\frac{\partial L}{\partial y^c_C} - \mathcal{C}^c_{ab}A^b_i y^a_C\frac{\partial L}{\partial y^c_C}\\
&\frac{\mathrm{d}}{\mathrm{d}t^A}\left(\frac{\partial L}{\partial y^a_A}\right) = \mathcal{C}^c_{ab}A^b_i y^i_C\frac{\partial L}{\partial y^c_C} - \mathcal{C}^c_{ab}y^b_C\frac{\partial L}{\partial y^c_C}\\
&0 = \frac{\partial y^i_A}{\partial t^B} - \frac{\partial y^i_B}{\partial t^A}\\
&0 = \frac{\partial y^c_A}{\partial t^B} - \frac{\partial y^c_B}{\partial t^A} - B^c_{ij}y^i_B y^j_A + \mathcal{C}^c_{ab}A^b_i y^i_B y^a_A + \mathcal{C}^c_{ab}y^b_A y^a_B.
\end{aligned}$$

If Q is a single point, that is $\bar{Q} = G$, then $T\bar{Q}/G = \mathfrak{g}$, the Lagrangian is a function $L : \overset{k}{\oplus}\mathfrak{g} \longrightarrow \mathbb{R}$ and the field equations reduce to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t^A}\left(\frac{\partial L}{\partial y^a_A}\right) = -c^c_{ab}y^b_C\frac{\partial L}{\partial y^c_C}\\
&0 = \frac{\partial y^c_A}{\partial t^B} - \frac{\partial y^c_B}{\partial t^A} + \mathcal{C}^c_{ab}y^b_A y^a_B,
\end{aligned}$$

a local form of the Euler–Poincaré equations (see, for instance, [6] and [32]).

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